EE 302

Introduction to Linear Control Systems

Spring 1979-80

Course Outline

1. Introduction and basic concepts
2. Modelling of physical systems
   Examples covering electromechanical systems
3. Transfer functions and block diagrams. Simplification rules for block diagrams
4. Time Domain Considerations:
   Quantification of transient response to step inputs
5. Steady state response and error coefficients
6. Stability of Control Systems:
   a. Routh-Hurwitz Criterion
   b. Root-locus
   c. Nyquist theorem
7. Frequency Domain Considerations: Bode diagrams, polar plots, phase margin, gain margin
8. Introduction to feedback design; lag, lead, and lag-lead compensators

References

1. "Modern Control Engineering",
   K. Ogata, Prentice Hall
2. "Automatic Control Systems",
   B. Kuo, Prentice Hall
3. "Modern Control Systems",
   R. Dorf, Addison-Wesley

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$y(t)$: output
$x(t)$: input

$x(t) \rightarrow$ [Syst.] $\rightarrow y(t)$

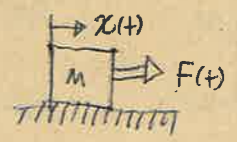
We can use n number of first order equations or nth order one diff. equation in order to represent a system.

Something about mechanical systems:

Mechanical translational systems:

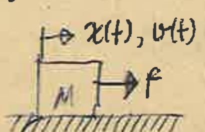
(1) Mass

$f(t)$: force
$x(t)$: displacement
$v(t)$: $\frac{dx(t)}{dt}$ = Velocity



Newton's law: $F(t) = M\frac{dv(t)}{dt}$ $(f = ma)$

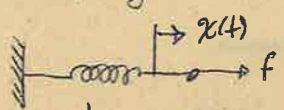
(2) Friction



$\rightarrow$ B : friction coefficient

$$M \cdot \frac{dv(t)}{dt} = F - \underbrace{B \cdot v(t)}_{\text{friction force}}$$

(3) Spring



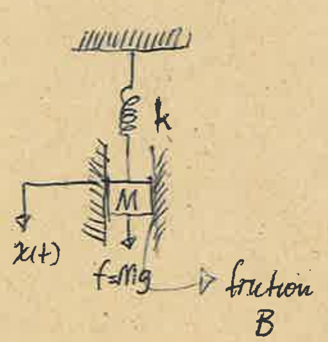
k = spring constant

$$\boxed{f(t) = k\,x(t)}$$

$$k\frac{dx(t)}{dt} = \frac{df(t)}{dt} \rightarrow \left\{ k\,v(t) = \frac{df(t)}{dt} \right\}$$

terminal equation of the spring.

Example :



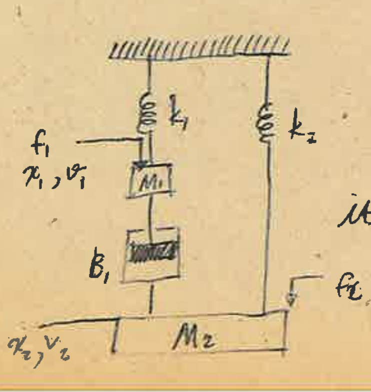
$$M\frac{dv(t)}{dt} = \overset{f}{Mg} - Bv(t) - k\,x(t)$$

$$M\frac{d^2v(t)}{dt} = \dot{f} - B\dot{v} - kv$$

Calculate the transfer funct. of this system.

input f
output v

Example :



inputs : f_1, f_2
outputs : v_1, v_2

it is a multi variable function.

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix} \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix}$$

$\hookrightarrow$ our aim is finding the entries of this matrix

$$M_1 \frac{d^2 x_1(t)}{dt^2} = f_1 - k_1 x_1(t) - B_1 \overset{(\dot{x}_1 - \dot{x}_2)}{(v_1 - v_2)}$$

relative velocity

pay attention

$$M_2 \frac{d^2 x_2(t)}{dt^2} = f_2 - k_2 x_2(t) - B(\dot{x}_2 - \dot{x}_1)$$

$$(M_1 s^2 + Bs + k_1) X_1(s) - Bs X_2(s) = F_1(s)$$

$$(M_2 s^2 + Bs + k_2) X_2(s) - Bs X_1(s) = F_2(s)$$

in matrix form:

$$\underbrace{\begin{bmatrix} M_1 s^2 + Bs + k_1 & -Bs \\ -Bs & M_2 s^2 + Bs + k_2 \end{bmatrix}}_{T = [\ ]^{-1}} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix}$$

- 4^{th} order.

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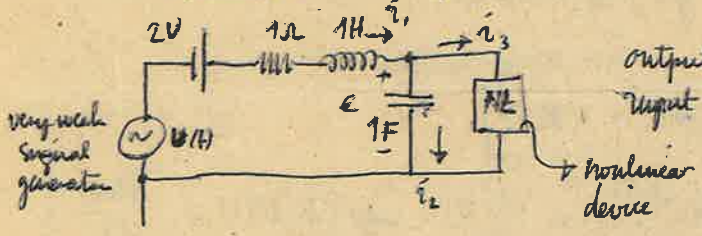
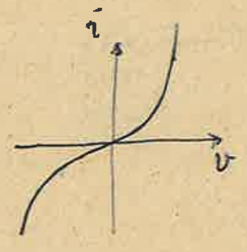
Linearization examples:



output : i_3
input v(t)

$$i_3 = f(v) \doteq v^3$$



$$v(t) \overset{\text{minus}}{+} 2 - i_1 - \frac{di_1}{dt} = e$$

$$i_1 = i_2 + i_3 = \frac{de}{dt} + e^3$$

$$y(t) = e^3(t)$$

$$\frac{di_1}{dt} = i_1 - e + 2 + v(t) = 0 \qquad v^{\circ} = 0$$

$$\frac{de}{dt} = i_1 - e^3$$

$$-i_1 - e + 2 = 0$$

$$i_1 - e^3 = 0$$

$$e^3 = -e + 2 \left| \begin{array}{l} e^3 + e - 2 = 0 \\ e^{\circ} = 1 \\ i_3^{\circ} = 1 \end{array} \right.$$

when we apply d.c source:

approximation: an R + a voltage source

$$\frac{d}{dt}\begin{pmatrix}\delta i_1\\ \delta e\end{pmatrix} = \begin{pmatrix}1 & -1\\ 1 & -3\end{pmatrix}\begin{pmatrix}\delta i_1\\ \delta e\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix}\delta U(t)$$

$y(t) = e^3$

$$\delta y(t) = (0 \;\; 3)\begin{pmatrix}\delta i_1\\ \delta e_1\end{pmatrix}$$

Higher order DE

Take Laplace Transform of all variables with zero initial conditions and solve for output in terms of input.

$$x_1^{(3)}(t) = 3x_1^{(1)}(t) + 2x_2^{(1)}(t) + x_1(t) + U(t)$$

$$s^3\hat{x}_1 = 3s\hat{x}_1 + 2s\hat{x}_2 + s\hat{x}_1 + \hat{U}$$

Mechanical Components:

a) translational motion

1. Spring

Force $= -K(x - x_0)$

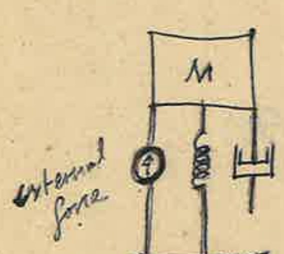
2. Mass

Force $= -M \cdot \underbrace{a}_{\frac{dv}{dt}}$

3. Friction element.

Force $= -B \underbrace{\frac{dx}{dt}}_{\text{velocity}}$

Example:



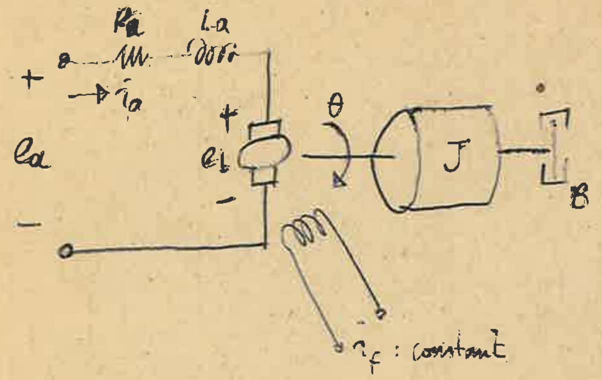
$$f(t) - kx - B\frac{dx}{dt} = M\frac{d^2x}{dt^2}$$

(gravity neglected)

$$\hat{f} - k\hat{x} - Bs\hat{X} = s^2 M\hat{X}$$

$$\hat{X}(s) = \frac{1}{Ms^2 + Bs + K}\hat{f}(s)$$

Armature Controlled d.c motors:



e_a : applied voltage (input of the system)

θ : angular displacement of the motor shaft

e_b : back e.m.f.

J : equivalent moment of inertia of the motor + load.

B : equivalent viscous friction coefficient

ψ : air gap flux

$\tau \propto i_a \psi$

$\psi \propto i_f$

$\tau = \underbrace{K_1 \cdot K_f}_{K} i_a \underset{\sim}{i_f}$

$\boxed{\tau = K i_a}$ (1)

$\boxed{e_b = K_b \omega}$ (2) $= K_b \dfrac{d\theta}{dt}$

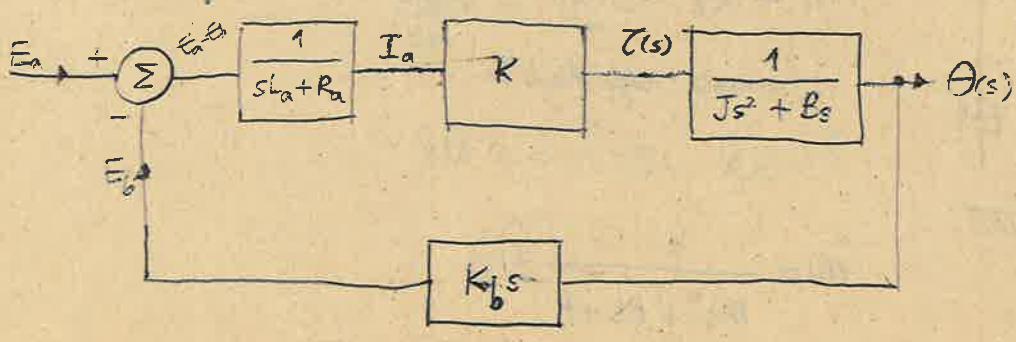
$$\boxed{R_a i_a + L_a \frac{di_a}{dt} + e_b = e_a} \quad (3)$$

$$\boxed{\tau = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt}} \quad (4)$$

Taking the Laplace transforms of equations:

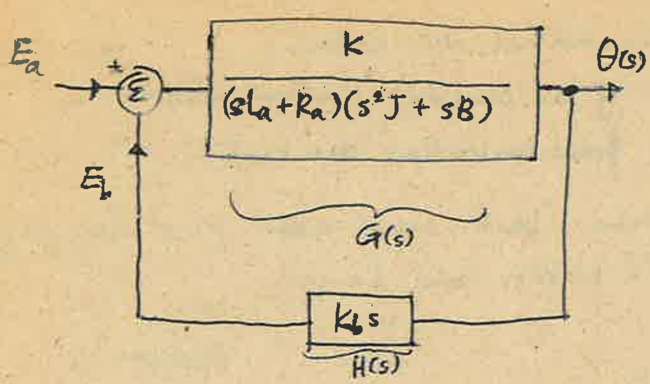
$$\boxed{\begin{aligned} &\tau(s) = K \cdot I_a(s) \\ &E_b(s) = K_b s \theta(s) \\ &L_a s I_a(s) + R_a I_a(s) + E_b(s) = E_a(s) \\ &T(s) = J s^2 \theta(s) + B s \theta(s) \end{aligned}}$$

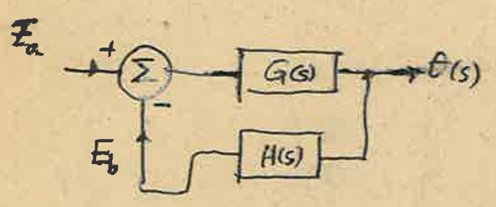
$$I_a = \frac{1}{sL_a + R_a}(E_a - E_b)$$



BLOCK DIAGRAM OF ARMATURE CONTROLLED D.C MOTOR

Equivalent block diagram





$E_b = \theta H(s)$

$$\theta(s) = G(s)\, e$$

$$e = E_a - E_b$$

$$= E_a - \theta(s) \cdot H(s)$$

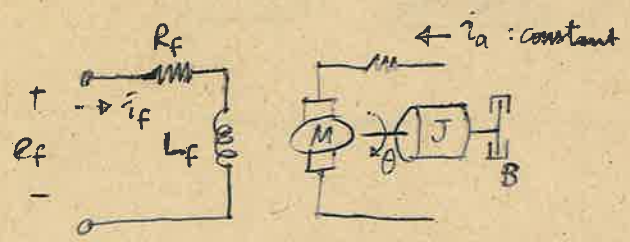
$$\theta(s) = G(s)\left[E_a - H(s)\,\theta(s)\right]$$

$$= G(s) E_a - G(s) H(s)\, \theta(s)$$

$$\left[1 + G(s)H(s)\right]\theta(s) = G(s) E_a$$

$$\frac{\theta(s)}{E_a(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Field Controlled d.c Motors :



e_f : applied voltage

i_a : armature current (constant)

for field controlled motor:

$$\tau = K_1 \psi\, i_a$$

↳ airgap flux

$$\psi \propto i_f \rightarrow = K i_f$$

$$\tau = \underbrace{K_2}_{K_1 K i_a} i_f$$

$$\boxed{\tau = K_2 i_f} \quad (1)$$

$$\boxed{L_f \frac{di_f}{dt} + R_f i_f = e_f} \quad (2)$$

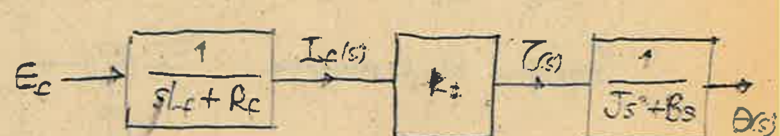
$$\boxed{\tau = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}} \quad (3)$$

Taking the Laplace Transforms:

$$\tau(s) = K_2\, I_f(s) \qquad (1)$$

$$(sL_f + R_f) I_f(s) = E_f(s) \qquad (2)$$

$$\tau(s) = (Js^2 + Bs)\,\theta(s) \qquad (3)$$



$$\frac{\theta(s)}{E_f(s)} = \frac{K_2}{(sL_f + R_f)(Js^2 + Bs)}$$

{ open loop system }

In Homework: $\frac{\theta_L(s)}{\cancel{E_f(s)}} \Rightarrow \frac{\theta_L(s)}{E(s)}$

Comparison of two systems:

① Amplifiers are simpler in field controlled d.c motors.

② I_a = constant ⇒ the requirement of a constant current source ... disadvantage of field controlled d.c motors.

③ In armature controlled d.c motors back e.m.f acts as a damping

④ Time constants of field controlled d.c motors are larger.

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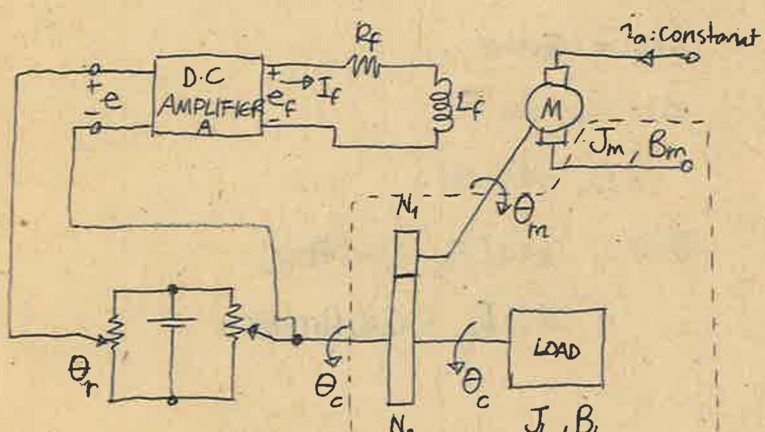
Example: (Position Control system)



$e = K_s(\theta_r - \theta_c)$

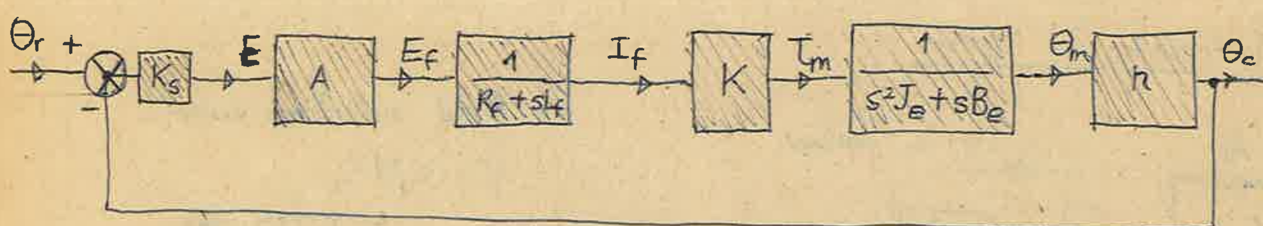
θ_r : input

θ_c : output

$$G(s) = \frac{\theta_c(s)}{\theta_r(s)}$$

$\{ \tau_m = K i_f \}$ generated by motor.

Equivalent block diagram:



$$n = \frac{N_1}{N_2} \qquad \theta_c = \frac{N_1}{N_2}\theta_m$$

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} + T_1$$

$$T_2 = \frac{1}{n} T_1 \Rightarrow J_L \frac{d^2\theta_c}{dt^2} + B_L \frac{d\theta_c}{dt} = J_L n \frac{d^2\theta_m}{dt^2} + B_L n \frac{d\theta_m}{dt} = \frac{1}{n} T_1$$

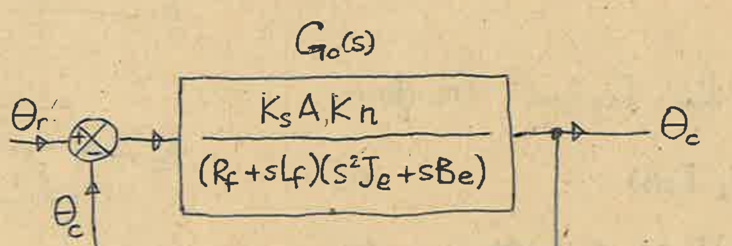
$$T_m = J_m \ddot{\theta}_m + B_m \dot{\theta}_m + n^2 J_L \ddot{\theta}_m + n^2 B_L \dot{\theta}_m$$

$$T_m = (J_m + n^2 J_L)\ddot{\theta}_m + (B_m + n^2 B_L)\dot{\theta}_m$$

$$J_e = J_m + n^2 J_L$$

$$B_e = B_m + n^2 B_L$$

Simplified block diagram:



$$\frac{\theta_c(s)}{\theta_r(s)} = \frac{G_o(s)}{1 + G_o(s)}$$

$\theta_c = G_o(s)\theta_e$

$\theta_e = \theta_r - \theta_c$

$\theta_c = G_o(s)(\theta_r - \theta_c)$

LINEARIZATION :

$f(x) \qquad x = x_0 + \Delta x$

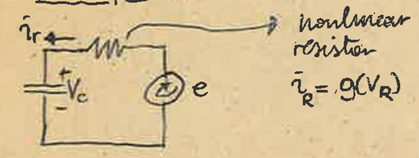
Taylor expansion : $f(x) = f(x_0+\Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2} f''(x_0)\Delta x^2 \ldots\ldots$ (very small)

$$\boxed{f(x) \cong f(x_0) + f'(x_0)\Delta x}$$

for multivariable function :

$$f(x_1, x_2, \ldots x_n) \cong f(x_1^\circ, x_2^\circ, \ldots x_n^\circ) + \frac{\partial f(x_1^\circ, x_2^\circ, \ldots x_n^\circ)}{\partial x_1}\Delta x_1 + \frac{\partial f(x_1^\circ, x_2^\circ \ldots x_n^\circ)}{\partial x_2}\Delta x_2 + \ldots + \frac{\partial f(x_1^\circ \ldots x_n^\circ)}{\partial x_n}\Delta x_n$$

Example :



$$C\frac{dV_c}{dt} = i_c = i_R = i = g(V_R) = g(e - V_c)$$

$$C\frac{dV_c}{dt} = g(e - V_c)$$

Let $C = 1$ and $g(x) = x^3 - x$

$$\frac{dV_c}{dt} = (e-V_c)^3 - (e - V_c)$$

Equilibrium points are calculated by equating $\dot{V}_c = 0$

$$(e-V_c)^3 - (e-V_c) = 0$$

$(e-V_c)\left[(e-V_c)^2 - 1\right] = 0 \qquad e = E \text{ (constant)} \qquad V_c = E \;, \; (E-V_c)^2 = 1$

$E = \pm V_c + 1 \qquad V_c = E+1, \quad V_c = -E+1$ equilibrium points

when we change the source voltage very weakly around E; if we linearize the function around the equilibrium points, we can find the transfer functions.

$$\frac{dV_c}{dt} = (e-V_c)^3 - (e-V_c) = f(e, V_c) \qquad e = E + \Delta e$$

first equilibrium point : $V_c = E \qquad (e = E)$

$$f(e, V_c) \cong f(E,E) + \frac{\partial f(E,E)}{\partial e}\Delta e + \frac{\partial f}{\partial V_c}\cdot\Delta V_c$$

$$\frac{d}{dt}(E + \Delta V_c) = f(e, V_c) = 0 + \left[3(e-V_c)^2 - 1\right]\Big|_{\substack{e=E \\ V_c=E}}\Delta e + \left[-3(e-V_c)^2 + 1\right]\Delta V_c\Big|_{\substack{e=E \\ V_c=E}} =$$

$$\boxed{\frac{d}{dt}\Delta V_c = -\Delta e + \Delta V_c}$$ this is the linearized diff eq. around the equilibrium point.

for the second equilibrium point :

$e = E + \Delta e$

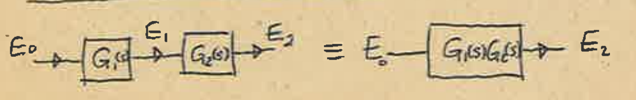
$V_c = (E-1) + \Delta V_c$

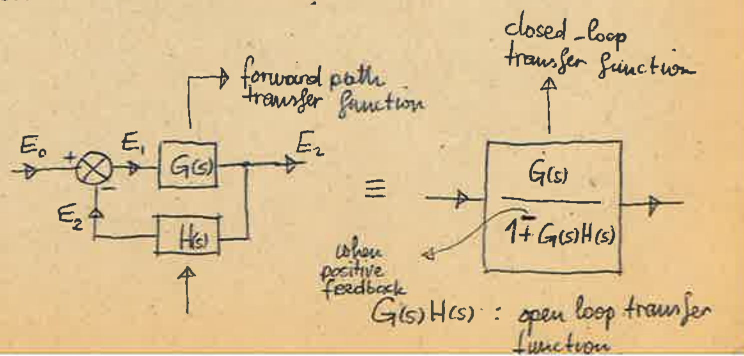
$$\frac{d\Delta V_c}{dt} = f(E, E-1) + \left\{3\left[e - V_c\right]^2 - 1\right\}\Big|_{E, E-1}\Delta e + \left[-3(e-V_c)^2 + 1\right]\Big|_{E, E-1}\Delta V_c$$

$$= \left[3(\cancel{E} - \cancel{E} + 1)^2 - 1\right]\Delta e + \left[-3(\cancel{E} - \cancel{E} + 1)\right]\Delta V_c$$

$$\boxed{\frac{d\Delta V_c}{dt} = 2\Delta e - 2\Delta V_c}$$ linearized diff. eq. around second equilibrium point. "different than the other one"

REDUCTION OF BLOCK DIAGRAMS :

① 

② 

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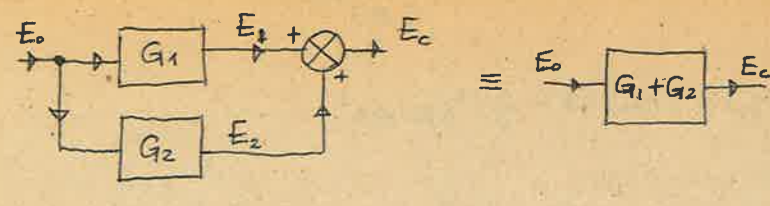

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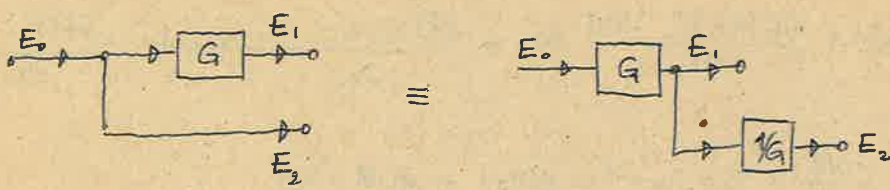

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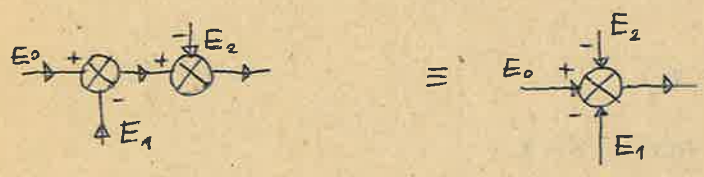
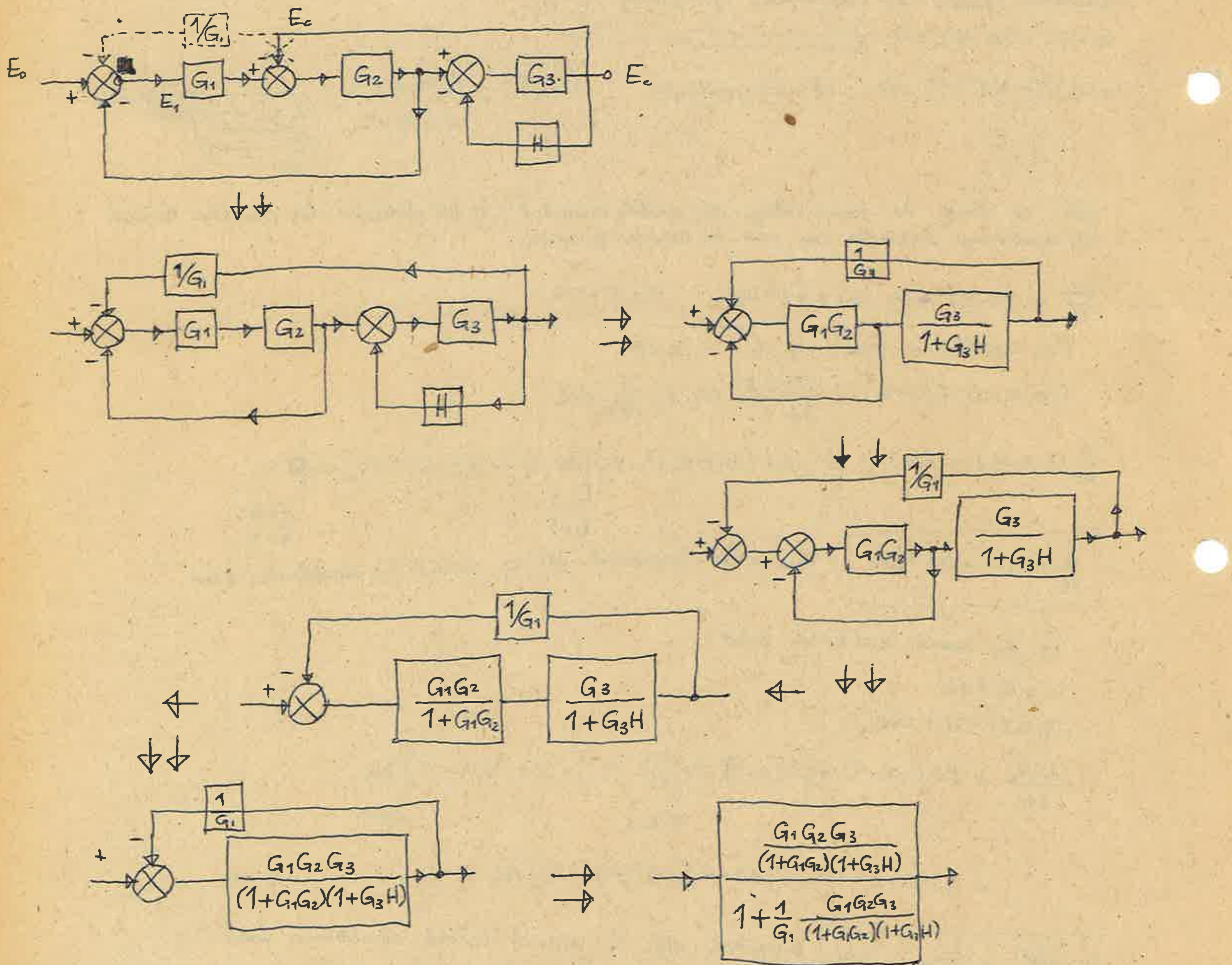
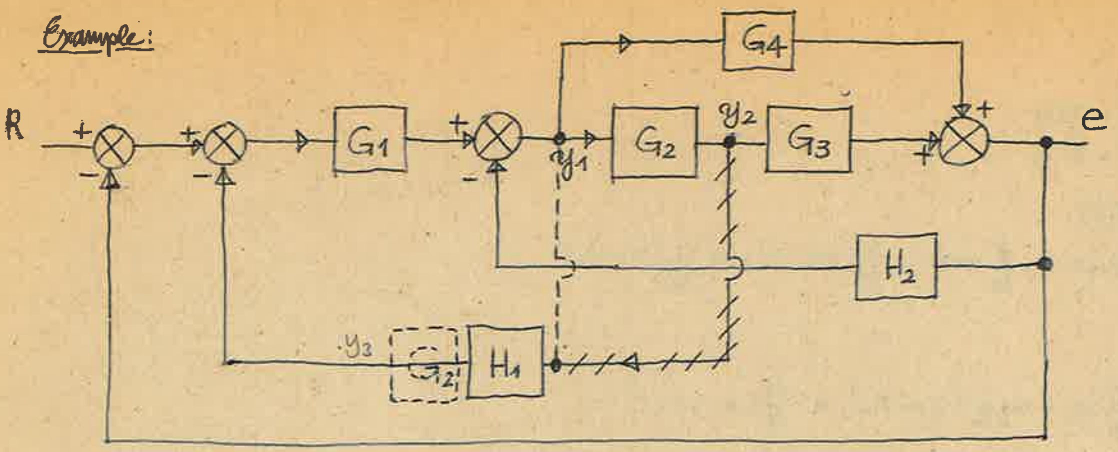


Example : (Reduction of block diagrams)



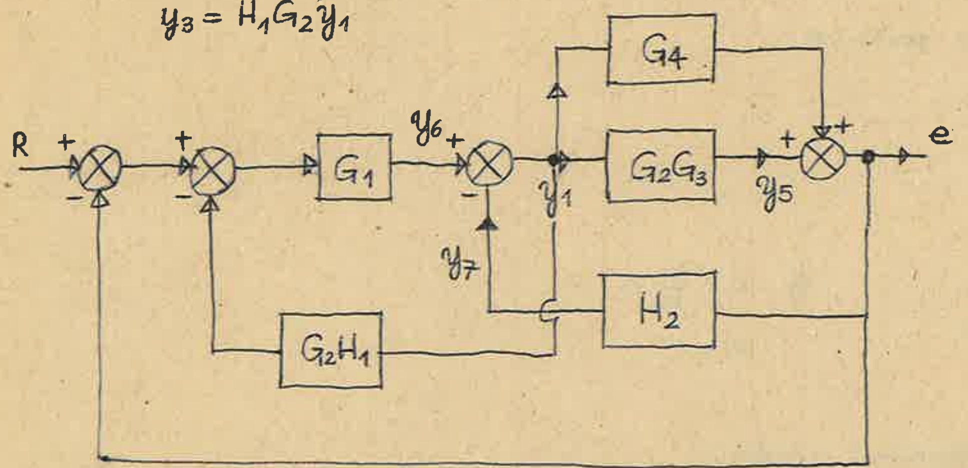
Example:



$y_3 = H_1 y_2$

$y_2 = G_2 y_1$

$y_3 = H_1 G_2 y_1$



$$e = \underbrace{y_1 G_4}_{y_4} + \underbrace{y_1 G_2 G_3}_{y_5} = (G_4 + G_2 G_3) y_1$$

Result :

$$\frac{e(s)}{R(s)} = \frac{G_1G_4 + G_1G_2G_3}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_4 + G_1G_2G_3}$$

TRANSIENT RESPONSE :

The subjects included:

- Stability
- Steady state error
- First order systems
- Transient response for ;
  a) Unit step function
  b) Unit ramp function.

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HW1 PROB 5 :

nonlinear spring : $f = k_1 x - k_2 x^2$

$$M\ddot{x} = -B\dot{x}(t) - k_1(x_1 - x_u) + k_2(x - x_u)^2$$

$$x_u(t) \equiv 0$$

defined :

$$x_1(t) = x(t)$$
$$x_2(t) = \dot{x}(t)$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = \ddot{x}(t) = -\frac{B}{M}x(t) - \frac{k_1}{M}(x - x_u) + \frac{k_2}{M}(x - x_u)^2$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{B}{M}x_2 - k_1/M\,(x_1 - x_u) + \frac{k_2}{M}(x_1 - x_u)^2$$

$$\dot{x} = \dot{x}_1 = 0 \quad \Big| \quad x_2 = 0$$
$$\ddot{x} = \dot{x}_2 = 0 \quad \Big| \quad -\frac{k_1}{M}x_1 + \frac{k_2}{M}x_1^2 = 0$$

from here equilibrium points are :

$$x_1(k_2 x_1 - k_1) = 0$$
$$x_1 = 0$$
$$x_1 = \frac{k_1}{k_2}$$

① $x_1 = 0 \Rightarrow x = 0$
$x_2 = 0 \Rightarrow \dot{x} = 0$

② $x_1 = \frac{k_1}{k_2}x$
$x_2 = 0$

Linearising the differential equation:

$$[0 + \delta u] \longrightarrow \begin{bmatrix} \underset{\text{changes}}{\delta x_1} + \overset{\text{equilibrium}}{x_{1e}} \\ \delta x_2 + x_{2e} \end{bmatrix}$$

At the first equilibrium point : $x_{1e} = 0$, $x_{2e} = 0$

$$\dot{x}_1 = x_2$$

① $\delta\dot{x}_1 = \partial x_2$

[redacted]

$$f(x_1, x_2, x_u) = -\frac{B}{M}x_2 - \frac{k_1}{M}(x_1 - x_u) + \frac{k_2}{M}(x_1 - x_u)^2$$

$$f(x_{1e} + \delta x_1,\ x_{2e} + \delta x_2,\ x_{ue} + \delta x_u) \cong \left(-\frac{k_1}{M} + \frac{k_2}{M}(x_1 - x_u)\right)\Big|_{eq} \cdot \delta x_1 - \frac{B}{M}\delta x_2 + \left(\frac{k_1}{M} - \frac{2k_2}{M}(x_1 - x_u)\right)\Big|_{eq}\delta x_u$$

we are calculating the partial derivatives

there are 3 variables.

$$\cong -\frac{k_1}{M}\delta x_1 - \frac{B}{M}\delta x_2 + \frac{k_1}{M}\delta x_u$$

$$\delta\dot{x}_1 = \delta x_2$$
$$\delta\dot{x}_2 = -\frac{k_1}{M}\delta x_1 - \frac{B}{M}\delta x_2 + \frac{k_1}{M}\delta x_u$$

$$\delta x_u \rightarrow U(s)$$
$$\delta x_1 = \delta x \rightarrow X(s)$$

$$\frac{X(s)}{U(s)} = ? \ :$$

<u>Method I :</u>

$$\delta\ddot{x}_1 = \delta\dot{x}_2 = -\frac{k_1}{M}\delta x_1 - \frac{B}{M}\delta\dot{x}_1 + \frac{k_1}{M}\delta x_u$$

$$s^2 X_1(s) = -\frac{k_1}{M}X_1(s) - \frac{B}{M}sX_1(s) + \frac{k_1}{s}U(s)$$

$$\frac{X_1(s)}{U(s)} = ?$$

<u>Method II</u>

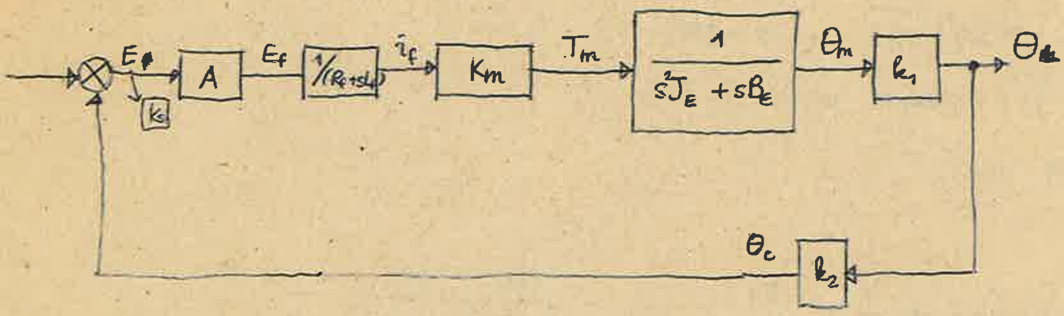
$$sX_1(s) = X_2(s)$$
$$sX_2(s) = \frac{k_1}{M}X_1(s) - \frac{B}{M}X_2(s) + \frac{k_1}{M}U(s)$$
$$\frac{X_1(s)}{U(s)} = ?$$

For the second equilibrium point :

$\delta\dot{x}_1 = \delta x_2$

$\delta\dot{x}_2 = \left(-\frac{k_1}{M} + 2\frac{k_2}{M}\frac{k_1}{k_2}\right)\delta x_1 - \frac{B}{M}\delta x_2 + \left(\frac{k_1}{M} - \frac{2k_2}{M}\frac{k_1}{k_2}\right)\delta x_1$

PROB 4 :



$N_1\theta_m = N_2\theta_2$

$\theta_2 N_3 = \theta_L N_4$

$\theta_L = \underbrace{\frac{N_3}{N_4}\cdot\frac{N_1}{N_2}}_{k_1}\theta_m \qquad J_e = J_m + k_1^2 J_L$

$B_e = B_m + k_1^2 B_L$

IMPULSE RESPONSE OF FIRST ORDER SYSTEM :

$G(s) = \frac{1}{Ts+1}$

$u(t) = \delta(t) \longrightarrow 1$

$Y(s) = \frac{1}{Ts+1} = \frac{1/T}{s + 1/T}$

$y(t) = \frac{1}{T}e^{-t/T}$

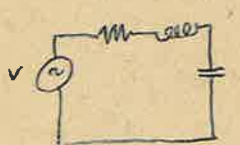
① Ramp response : $t - T + Te^{-t/T}$) derivative — steady state error : T

② Step response : $1 - e^{-t/T}$) derivative — 0

③ Impulse response : $\frac{1}{T}e^{-t/T}$ — 0

SECOND ORDER SYSTEMS :

Example (Series RLC network)



$\frac{V_c(s)}{V(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + sL + R} = \frac{1}{s^2LC + RCs + 1}$

$\frac{V_c}{V} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

ω_n : undamped natural frequency.

ξ : damping ratio

$\delta = \xi\omega_n$ = attenuation

Roots of the polynomial: $s^2 + 2\xi\omega_n + \omega_n^2$

$$S_{12} = -\xi\omega_n \pm\sqrt{\omega_n^2\xi^2 - \omega_n^2}$$

$$S_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{-1+\xi}$$

① Underdamped case: $0<\xi<1$

$$\therefore\ S_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2} = -\sigma + j\omega_d$$

$$S_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2} = -\sigma - j\omega_d$$

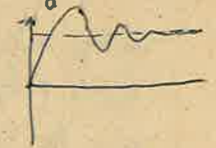
$\omega_d = \omega_n\sqrt{1-\xi^2}$: damped natural frequency

Response to unit step input:

$$U(s) = \frac{1}{s}$$

$$Y(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\,\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$y(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t - \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right] \quad t\geq 0 \quad \rightarrow$$



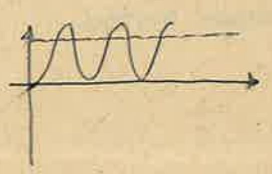
$y(\infty) = 1$

steady state error $= 0$ (s.s.e)

if $\xi = 0$

$\omega_d = \omega_n\sqrt{1-\xi^2} = \omega_n$

$y(t) = 1 - (\cos\omega_n t) \longrightarrow$



② Critically damped case: $(\xi = 1)$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = s^2 + 2\omega_n s + \omega_n^2 = (s+\omega_n)^2$$

Unit step response

$$Y(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

$$y(t) = 1 - e^{-\omega_n t}\left[1+\omega_n t\right]$$

steady state error $= 0$

③ Overdamped case $(\xi > 1)$

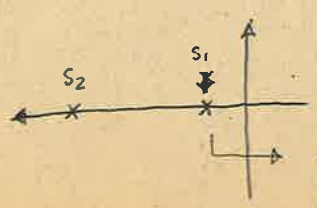
$$s^2 + 2\xi\omega_n s + \omega_n^2 \Rightarrow S_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2-1}$$

$$S_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2-1}$$

Step response

$$Y(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{1}{s} + \frac{\omega_n^2}{s_1(s_1-s_2)}\,\frac{1}{(s+s_1)} + \frac{\omega_n^2}{s_2(s_2-s_1)}\,\frac{1}{(s+s_2)}$$

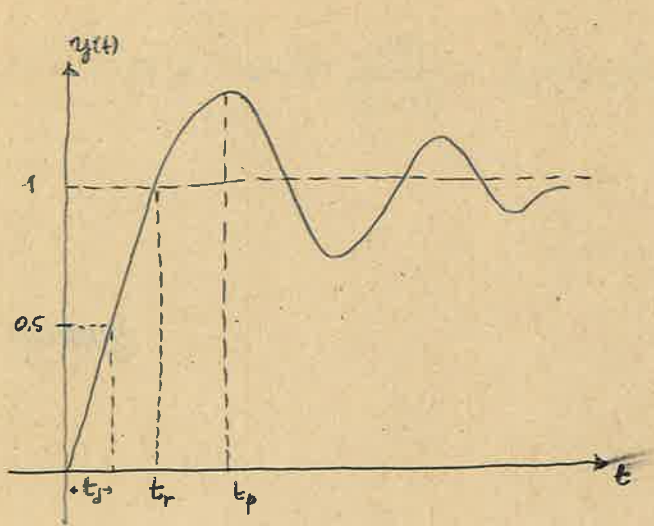
$$y(t) = 1 + \frac{\omega_n^2}{(s_1-s_2)}\left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right]$$



important for stability analysis (the root closer to the origin)

$$T(s) = \frac{\omega_n^2}{s^2 + 2s\xi\omega_n + \omega_n^2} \qquad u(s) = \frac{1}{s}$$

$$y(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]$$



delay time (t_d) : time for which $y(t)$ reaches to half of its steady state value.

rise time (t_r) : 5% → 95%
10% → 90%
0% → 100%

peak time (t_p) : time required for reaching the first peak value or ~~maximum overshoot~~.

Maximum overshoot (M_p)

$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \%100$$

Settling time (t_s)

It is the time for which the $y(t)$ reaches the value that s.s.e less the 2 %

* If $0.4 \leq \xi \leq 0.8$ then the response is good.

CALCULATIONS :

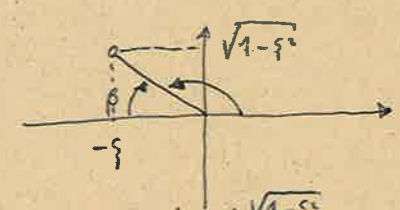
1) Rise time t_r

$$y(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

$$1 = y(t_r) = \cancel{1} - e^{-\xi\omega_n t_r}\underbrace{\left[\cos\omega_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_r\right]}_{\text{must be equal to zero}}$$

$$\cos\omega_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_r = 0$$

$$\tan\omega_d t_r = +\frac{\sqrt{1-\xi^2}}{-\xi}$$



$$\omega_d t_r = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \pi - \beta = \pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

$$\boxed{t_r = \frac{\pi - \beta}{\omega_d}}$$

$0 \leq \beta \leq \pi/2$ for large ω_n , t_r is small.

2. Peak time

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$y'(t) = 0 = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) - e^{-\zeta\omega_n t}\left(-\omega_d\sin\omega_d t + \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)\Bigg|_{t_p} = 0$$

$$\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d\sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t = 0$$

$$\therefore \sin\omega_d t_p = 0$$

$$\omega_d t_p = 0, \pi, 2\pi \; ----$$

$$t_p = \frac{\pi}{\omega_d}$$

21MAY80

3. Maximum overshoot : M_p

$$M_p = \frac{y(t_p) - y_{ss}}{y_{ss}} \times 100\,\%$$

$$y_{ss} = 1$$

$$t_p = \frac{\pi}{\omega_d}$$

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \quad t \geq 0$$

$$e(t) = u(t) - y(t) = e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

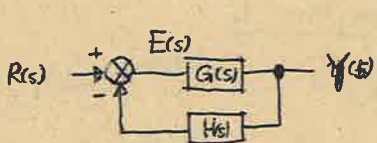
$$M_p = -e^{-\zeta\omega_n \pi/\omega_d}\left(\overbrace{\cos\cancel{\omega_d}\frac{\pi}{\cancel{\omega_d}}}^{-1} + \frac{\zeta}{\sqrt{1-\zeta^2}}\underbrace{\sin\cancel{\omega_d}\frac{\pi}{\cancel{\omega_d}}}_{0}\right)$$

$$= e^{-\zeta\frac{\omega_n\pi}{\omega_d}}$$

$$= e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}}$$

4. Settling time :

$$T_s \cong \frac{4}{\zeta\omega_n}$$ '4 time constant. error 2%



5. Steady State error

u(t) → unit step
unit ramp
parabola
sinusoidal

$$G(s)H(s) = \frac{K(a_m s^m - a_{m-1}s^{m-1} --- a_1 s + 1)}{s^N(b_n s^n + b_{n-1}s^{n-1} ---, b_1 s + 1)}$$

Type 0 if $N=0$
" 1 " $N=1$
" 2 " $N=2$

$$E(s) = R(s) - H(s)Y(s)$$

$$= R(s) - H(s)G(s)E(s)$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + H(s)G(s)}$$

* Step response : $R(s) = \frac{1}{s}$

$$E(s) = \frac{1}{1 + \frac{Kp(s)}{s^N q(s)}} \cdot \frac{1}{s} = \frac{s^N q(s)}{s^N q(s) + Kp(s)} \cdot \frac{1}{s}$$

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \longrightarrow N=0 \Rightarrow e_{ss} = \lim_{s \to 0} \cancel{s} \frac{q(s)}{q(s)+Kp(s)} \cdot \frac{1}{\cancel{s}} = \frac{q(0)}{q(0)+Kp(0)} = \frac{1}{1+K}$$

K : position error coefficient

$\longrightarrow N=1 \Rightarrow$

$$e_{ss} = \lim_{s \to 0} s \frac{sq(s)}{sq(s)+Kp(s)} \frac{1}{s} = \frac{0.1}{0.1+K.1} = 0$$

$\therefore$ for $N > 0 \Rightarrow e_{ss} = 0$.

K_p : static position error coefficient $= G(0)H(0)$

$N = 0 \Rightarrow K_p = K \Rightarrow e_{ss} = \frac{1}{1+K}$

$N > 0 \Rightarrow K_p = \infty \qquad e_{ss} = 0$

* Ramp input $r(t) = t \quad R(s) \Rightarrow \frac{1}{s^2}$

$N = 0$

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \frac{s}{s^2} \frac{q(s)}{q(s)+Kp(s)} = \infty$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{1+G(s)H(s)} \frac{\cancel{s}}{s^{\cancel{2}}} = \lim_{s \to 0} \frac{1}{s+sG(s)H(s)}$$

K_v = static velocity error coefficient $\lim_{s \to 0} sG(s)H(s)$

$N = 0 \Rightarrow K_v = 0 \Rightarrow e_{ss} = \frac{1}{K_v} = \infty$

$N = 1 \Rightarrow$

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} s \cdot \frac{Kp(s)}{sq(s)} = K$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}$$

$N = 2 \Rightarrow$

$$K_v = \lim_{s \to 0} s \frac{Kp(s)}{s^2 q(s)} = \infty \Rightarrow e_{ss} = 0$$

$N \geq 2 \Rightarrow K_v = \infty \quad \therefore e_{ss} = 0$

*** Unit parabola** $\quad r(t) = \frac{1}{2}t^2 \qquad R(s) = \frac{1}{s^3}$

$$E(s) = \frac{1}{1+G(s)H(s)}\,\frac{1}{s^3}$$

$$e_{ss} = \lim_{s\to 0} \frac{\cancel{s}}{1+G(s)H(s)}\,\frac{1}{\cancel{s^3}\,2} = \lim_{s\to 0}\frac{1}{s^2 + s^2 G(s)H(s)}$$

K_a : static acceleration error coefficient.

$N = 0 \Rightarrow$

$$K_a = \lim_{s\to 0} s^2 \frac{P(s)}{q(s)} K = 0$$

$$e_{ss} = \infty$$

$N = 1 \Rightarrow$

$$K_a = \lim_{s\to 0} s^{\cancel{2}} \frac{KP(s)}{\cancel{s}q(s)} = 0$$

$$e_{ss} = 0$$

$N=2 \Rightarrow \quad K_a = \lim_{s\to 0} s^2 \frac{KP(s)}{s^2 q(s)} = K$

$$\therefore e_{ss} = \frac{1}{K}$$

$N = 3 \Rightarrow \quad K_a = \lim_{s\to 0} s^2 \frac{KP(s)}{s^3 q(s)} = \infty$

$$e_{ss} = 0$$

$r(t) = t^i e^{s_0 t} \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$

$i=0$	$S_0 = 0$	$\Rightarrow$	1	unit step
$i=1$	$S_0 = 0$	$\Rightarrow$	t	unit ramp
$i=2$	$S_0 = 0$	$\Rightarrow$	t^2	parabola
$i=0$	$S_0 = j\omega$	$\Rightarrow$	$e^{j\omega t}$	sinusoidal input

$$Y(s) = T(s)R(s)$$

$$T(s) = \frac{(s-a_1)(s-a_2)\cdots(s-a_m)}{(s-b_1)(s-b_2)\cdots(s-b_n)} \qquad n \geq m \quad \text{stable system}$$

$$r(t) = e^{s_0 t} \qquad R(s) = \frac{1}{s-s_0}$$

$k_1 = T(s_0)$

$$Y(s) = \frac{1}{s-s_0}\,\frac{(s-a_1)(s-a_2)\cdots(s-a_m)}{(s-b_1)(s-b_2)\cdots(s-b_n)} = \frac{k_1}{s-s_0} + \frac{k_2}{s-b_1} + \cdots$$

$=$

$$\boxed{Y_{ss}(s) \cong \frac{T(s_0)}{s-s_0}}$$

* $r(t) =$ unit step $\Rightarrow S_0 = 0 \quad y_{ss}(t) = T(0)$

$$T(s) = \frac{1}{1+G(s)H(s)} \Rightarrow e_{ss} = \frac{1}{1+G(0)H(0)}$$

* $r(t) =$ unit ramp $\Rightarrow \left.\frac{de^{S_0t}}{dt}\right|_{S_0=0} = te^{S_0t} = t$

$$\therefore y_{ss}(t) = \left.\frac{d}{ds_0}\left(T(s_0)e^{s_0t}\right)\right|_{s_0=0} = \left.T'(s_0)e^{s_0t} + tT(s_0)e^{s_0t}\right|_{s_0=0} = T'(0) + tT(0)$$

$$T(s) = \frac{1}{1+\frac{KP(s)}{s^Nq(s)}}$$

$N=0$;

$$T(s) = \frac{1}{1+\frac{KP(s)}{q(s)}} = \frac{q(s)}{q(s)+KP(s)}$$

$$e_{ss}(t) = T'(0) + t\frac{1}{1+K} = \underbrace{\frac{q'(s)[q(s)+KP(s)] - [q'(s)+KP'(s)]q(s)}{(q(s)+KP(s))^2}}_{K_1} + t\frac{1}{1+K} = K_1 + t\frac{1}{1+K}$$

$N=1$;

$$e_{ss} = T'(0) + tT(0)$$

$$T(s) = \frac{1}{1+\frac{KP(s)}{sq(s)}} = \frac{sq(s)}{sq(s)+KP(s)}$$

$$T(0) = 0$$

$$T'(s) = \frac{[q(s)+sq'(s)][sq(s)+KP(s)] - [\frac{d}{ds}(sq(s)+KP(s)]sq(s)}{\{sq(s)+KP(s)\}^2}$$

$$T'(0) = \frac{\{1+0\}\cdot\{0+K1\} - \{\quad\}\cdot 0}{\{0+K\}^2} = \frac{K}{K^2} = \frac{1}{K}$$

$$e_{ss} = \frac{1}{K}$$

* response to parabolic input

$$\left.\frac{d^2}{ds_0^2}e^{S_0t}\right|_{S_0=0} = \left.\frac{d}{ds_0}(te^{s_0t})\right|_{S_0=0} = \left.t^2e^{S_0t}\right|_{S_0=0} = t^2$$

$$\therefore y_{ss}(t) = \left.\frac{d^2}{ds_0^2}\left(T(s_0)e^{s_0t}\right)\right|_{S_0=0}$$

$$\frac{d}{ds_0}\left(T'(s_0)e^{s_0t} + tT(s_0)e^{s_0t}\right) = T''(s_0)e^{s_0t} + tT'(s_0)e^{s_0t} + tT'(s_0)e^{s_0t} + t^2T(s_0)e^{s_0t}$$

$$t^2 \rightarrow y_{ss}(t) = T''(0) + 2tT'(0) + t^2T(0)$$

Another method:

Taylor expansion around $s=0$

$$\frac{1}{1+G(s)} = \frac{P(s)}{q(s)} = \frac{1}{k_1} + \frac{1}{k_2}s + \frac{1}{k_3}s^2 \dots$$

k_1 : dynamic position error coefficient
k_2 : dynamic velocity error coefficient
k_3 : dynamic acc. error coefficient

<u>Example</u>:

$$T(s) = \frac{1}{\cancel{1+}G(s)} = \frac{Ts+1}{Ts+1+K}$$

$$G(s) = \frac{K}{Ts+1} \qquad \text{type} = N=0$$

$$K_p = K \qquad K_v = 0$$

$$e_{ss} = \frac{1}{1+K} \qquad e_{ss} = \infty \Rightarrow T'(0) + tT(0)$$

$$e_{ss} = \frac{t}{1+K} + \left.\frac{T(Ts+1+K) - T(s+1)}{(Ts+1+K)^2}\right|_{s=0}$$

$$e_{ss}(t) = \frac{TK}{(1+K)^2} + t\frac{1}{1+K}$$

$$T(s) = \frac{Ts+1}{Ts+(K+1)}$$

$$1+Ts \;\Big|\; (1+K)+Ts$$

$$\frac{1}{1+K} + \frac{TK}{(1+K)^2}s + \frac{T^2K}{(1+K)^3}s^2 + \dots$$

$$\frac{-1 - \frac{T}{1+K}s}{\underbrace{\left(T - \frac{T}{1+K}\right)s}}$$

$$\frac{TK}{1+K}\cdot s + \frac{T^2K}{(1+K)^2}s^2$$

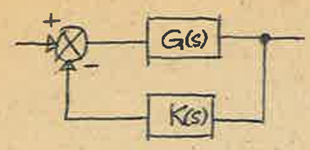
$$\frac{1}{k_1} = \frac{1}{1+K} \qquad k_1 = \frac{1+K}{1}$$

$$\frac{1}{k_2} = \frac{TK}{(1+K)^2} \qquad k_2 = \frac{(1+K)^2}{TK}$$

$$\frac{1}{k_3} = \frac{T^2K}{(1+K)^3}$$

28MAY80

example :



$$G(s) = \frac{3s^2+s}{s^3+2s+4} \qquad 1+G(s)K = 1+\frac{K(3s^2+s)}{s^3+2s+4}$$

$$= \frac{s^3+2s+4+3Ks^2+Ks}{s^3+2s+4}$$

$$q(s) = s^3+3Ks^2+(2+K)s+4$$

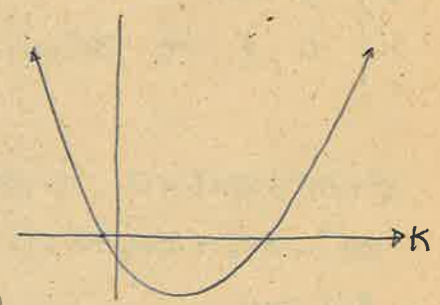
$$\begin{array}{c|cc} s^3 & 1 & (2+K) \\ s^2 & 3K & 4 \\ s & \frac{3K^2+6K-4}{3K} & 0 \\ 1 & 4 & \end{array}$$

For the system to be stable :

$$3K>0 \Rightarrow K>0$$

$$\frac{3K^2+6K-4}{3K} > 0$$

$$= \frac{(3K\ \ )(K\ \ )}{3K}$$



$$K > -1+\sqrt{\frac{21}{9}}$$

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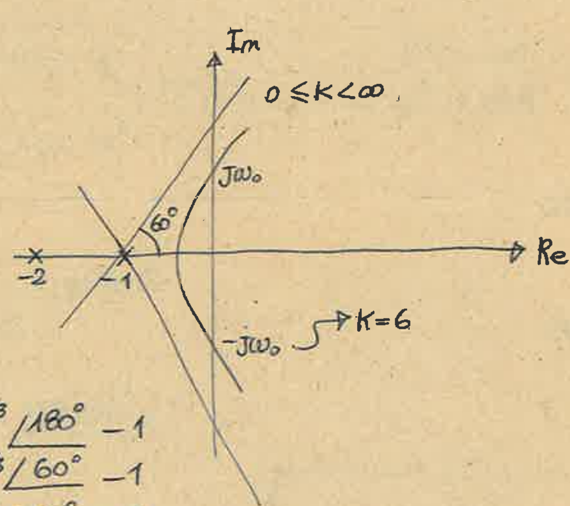
$$G(s) = \frac{K}{s(s+1)(s+2)}$$

$$s(s+1)(s+2)+K=0$$

$$(s+1)^3+K=0$$



$$(s+1)^3 = -K \rightarrow \quad s_1 = K^{1/3}\angle 180^\circ - 1$$

$$s_2 = K^{1/3}\angle 60^\circ - 1$$

$$s_3 = K^{1/3}\angle -60^\circ - 1$$

Calculation of the breakaway point :

$$G(s)H(s) = K\frac{P(s)}{q(s)} = -1$$

$$q(s)+Kp(s) = 0 \qquad \underbrace{\frac{d}{ds}q(s)}_{q'(s)} + K\underbrace{\frac{d}{ds}p(s)}_{p'(s)} = 0$$

$$K = \left.\frac{-q'(s)}{p'(s)}\right|_{s=s_b} \rightarrow \text{breakaway point}$$

$$\frac{q(s)\,p'(s)}{p'(s)} = \frac{q'(s)\,p(s)}{p'(s)} \qquad (s=s_b)$$

$$K = \frac{-q(s)}{p(s)} \qquad \frac{dK}{ds} = \frac{-q'(s)p(s)+q(s)p'(s)}{p(s)^2}$$

$$\frac{K}{s(s+1)(s+2)} = -1 \Rightarrow -K = s(s+1)(s+2)$$

$$-\frac{dK}{ds} = \frac{d}{ds}(s^3+3s^2+2s) = 3s^2+6s+2$$

$$s_{12} = \frac{-3\pm\sqrt{9-6}}{3} \quad \begin{cases} s_1 = \frac{-3-\sqrt{3}}{3} = -1.6\ \times \\ s_2 = \frac{-3+\sqrt{3}}{3} = -0.4\ \checkmark \end{cases}$$

$s(s+1)(s+2)+K=0$

$s^3+3s^2+2s+K=0 \Rightarrow 6-K>0$

$6>K$

(for the stability of the system)

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s & \frac{6-K}{3} & 0 \\ 1 & K & \end{array}$$

for $K=0$

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & 6 \\ s & 0 & 0 \\ 1 & & \end{array} \rightarrow 3s^2+6 \Rightarrow s^2=-2 \begin{cases} s_1 = j\sqrt{2} \\ s_2 = -j\sqrt{2} \end{cases}$$

$s^3+3s^2+2s+6=0$ $\quad s=j\omega_o$ is a root of this polynomial

$-j\omega_o^3-3\omega_o^2+2j\omega_o+6=0$

Real part:

$-3\omega_o^2+6=0$

$\omega_o^2=2 \Rightarrow \omega_o=\sqrt{2}$

Imaginary part:

$-j\omega_o^3+2j\omega_o=0$

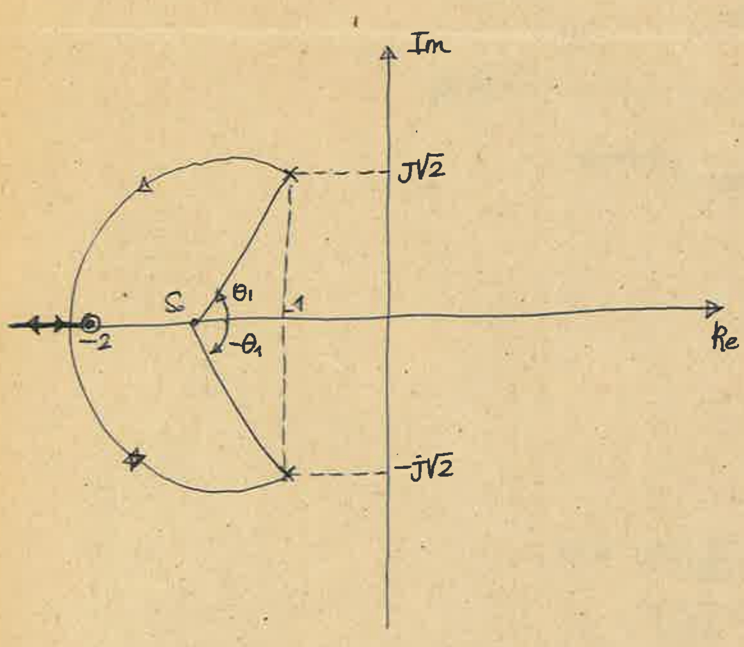
$\omega_o^2=2 \quad \omega_o=\sqrt{2}$

real part $\neq 0$

Example:

$G(s)H(s) = \dfrac{K(s+2)}{s^2+2s+3}$ $\quad$ zero $=-2$

poles $= S_{12} = -1 \pm \sqrt{1-3}$ $\begin{cases} -1+j\sqrt{2} \\ -1-j\sqrt{2} \end{cases}$



$G(s)H(s) = \dfrac{K(s_o+2)}{(s_o-s_1)(s_o-s_2)}$

$\angle G(s)H(s) = \angle s_o+2 - \angle s_o-s_1 - \angle s_o-s_2$

$= 0-\theta_1-(-\theta_1)=0$

$S_o \notin$ root locus.

$\angle G(s_o)H(s_o) = \pi-\theta-(-\theta)=\pi$

equation of the asymptot:

$S_o = K^\alpha\omega_o+z_o$ $\qquad s_2+2s+3+Ks+2K=0$

$(K^\alpha\omega_o+z_o)^2+2(K^\alpha\omega_o+z_o)+3+K(K^\alpha\omega_o+z_o)+2K=0$

$K^{2\alpha}\omega_o^2+2K^\alpha\omega_o z_o+2K^\alpha\omega_o+2z_o+3K^{\alpha+1}\omega_o+Kz_o+2K=0$
$+z_o^2$

$2\alpha=\alpha+1 \Rightarrow \alpha=1$

$K^2\omega_o^2+2K\omega_o z_o+z_o^2+2K\omega_o+2z_o+3K^2\omega_o+Kz_o+2K=0$

$\underset{\substack{\| \\ 0 \\ \Downarrow \\ \omega_o=-1}}{(K_2\omega_o^2+K^2\omega_o)} + K\underbrace{(2\omega_o z_o+z_o+2\omega_o+2)}_{\substack{\| \\ 0}}$

$-2z_o+z_o-2+2=0$

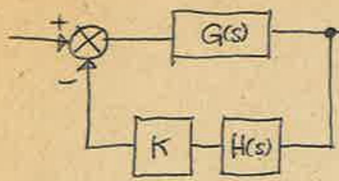
$z_o=0$

$\angle G(s_o)H(s_o) \cong \tan^{-1}\sqrt{2}-\theta-90^\circ=-180^\circ$

$\theta = \tan^{-1}\sqrt{2}+90^\circ$

$=155^\circ$

$\frac{P(s)}{q(s)} = G(s)H(s)$ is given



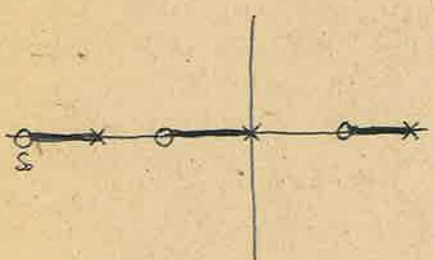
closed loop system

$$G_{cl}(s) = \frac{G(s)}{1 + K\,G(s)\,H(s)}$$

$\therefore$ poles of closed loop system satisfies the equation

$$1 + K\underbrace{G(s)H(s)}_{\frac{P(s)}{q(s)}} = 0 \quad \Rightarrow \quad q(s) + KP(s) = 0$$

① Plot the zeros and poles of the open loop system (ie. the roots of the polynomials $P(s)$, $q(s)$ respectively)
② Let s_0 be a real number. Then s_0 belongs to root locus if and only if the total number of zeros and poles on the real line, which are at the right of s_0 is odd.



③ Let $s = K^{\alpha}\omega_0 + z_0$

$$q(K^{\alpha}\omega_0 + z_0) + Kp(K^{\alpha}\omega_0 + z_0) = 0$$

$$q(s) = s^n + a_1 s^{n-1} + \dots + a_n$$

$$p(s) = s^m + b_1 s^{m-1} + \dots + b_m$$

$$\Rightarrow q(s) = (K^{\alpha}\omega_0 + z_0)^n + a_1(K^{\alpha}\omega_0 + z_0)^{n-1} + \dots + a_n$$

$$Kp(s) = K(K^{\alpha}\omega_0 + z_0)^m + b_1(K^{\alpha}\omega_0 + z_0)^{m-1} + \dots + b_m$$

$$q(s) + Kp(s) = \underbrace{K^{n\alpha}\omega_0^n + nK^{(n-1)\alpha}\omega_0^{n-1}z_0 + \dots}_{(K^{\alpha}\omega_0 + z_0)^n}$$

$$+ a_1K^{\alpha(n-1)}\omega_0^{n-1} + a_1(n-1)K^{\alpha(n-2)}\omega_0^{n-2}z_0 + \dots$$

$$+ K^{m\alpha+1}\omega_0^m + mK^{(m-1)\alpha+1}\omega_0^{m-1}z_0 + \dots$$

$$n\alpha = m\alpha + 1 \quad \Rightarrow \quad \boxed{\alpha = \frac{1}{n-m}}$$

$$\Rightarrow s = K^{\frac{1}{n-m}}\omega_0 + z_0$$

$$nK^{(n-1)\alpha}\omega_0^{n-1}z_0 + a_1K^{(n-1)\alpha}\omega_0^{n-1} + mK^{(m-1)\alpha+1}\omega_0^{m-1}z_0 + b_1K^{\alpha(m-1)+1}\omega_0^{\overset{\text{minus}}{m+1}} = 0$$

$$(m-1)\alpha + 1 = (n-1)\alpha$$

$$(m-1)\frac{1}{n-m} + 1 = (n-1)\frac{1}{n-m}$$

$$\frac{1}{n-m}(m - \cancel{1} - n + \cancel{1}) = -1$$

$$n\omega_0^{n-1}z_0 + a_1\omega_0^{n-1} + m\omega_0^{m-1}z_0 + b_1\omega_0^{m-1} = 0$$

$$z_0(n\omega_0^{n-1} + m\omega_0^{m-1}) + a_1\omega_0^{n-1} + b_1\omega_0^{m-1} = 0$$

$$\boxed{\omega_o^n + \omega_o^m = 0} \qquad \omega_o^m(\omega_o^{n-m}+1) = 0.$$

$$\boxed{\omega_o = 1\angle\theta} \qquad \boxed{(n-m)\theta = 180°}$$

$$\omega_o^{m-1}(z_o(n\omega_o^{n-m}+m) + a_1\omega_o^{n-m} + b_1) = 0$$

$$z_o(-n+m) + (b_1 - a_1) = 0 \qquad \boxed{z_o = \frac{b_1 - a_1}{n-m}}$$

$n-m$: # of open loop poles − # of open loop zeros

$-a_1 = \sum$ open loop poles

$-b_1 = \sum$ open loop zeros

$$z_o = \frac{\sum \text{open loop poles} - \sum \text{open loop zeros}}{\text{\# of open loop poles} - \text{\# of open loop zeros}}$$

$$(n-m)\theta = \pi(2k+1)$$

$$s = K^{1/(n-m)}\angle\theta + z_o$$

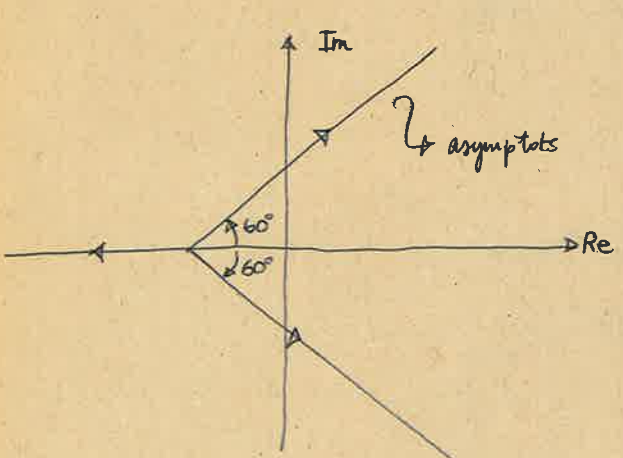
Example : $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$

# of open loop zeros : $0 = m$
# of open loop poles : $3 = n$ $\qquad n-m = 3$

$\theta_1 = 60°$
$\theta_2 = -60°$
$\theta_3 = 180°$

$$z_o = \frac{(0-1-2) - -}{3} = \frac{-3}{3} = -1$$

$s = K^{1/3}\angle 60° - 1$
$s = K^{1/3}\angle -60° - 1$

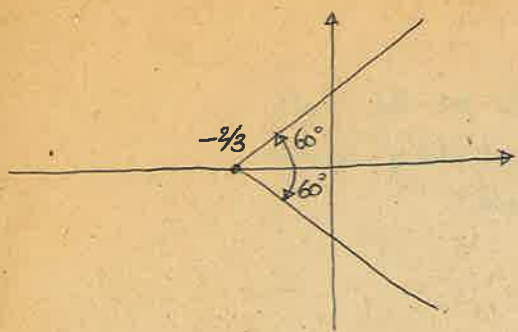


Example : $G(s)H(s) = \dfrac{K(s+1)}{s(s-1)(s^2+4s+6)}$

# of zeros $= m = 1$ $\qquad n-m = 3 \qquad \theta_1 = 60° \quad \theta_2 = -60° \quad \theta_3 = 180°$
# of poles $= n = 4$

zero : -1
poles : $0,\ 1\ ;\ -2+\sqrt{4-16}\ ,\ -2-\sqrt{4-16}$ $\qquad \sum$ of poles $= 0+1-2+\cancel{\sqrt{-12}} - 2 - \cancel{\sqrt{-12}}$

$$z_o = \frac{\overbrace{-3}^{\sum \text{poles}} - \overbrace{(-1)}^{\sum \text{zeros}}}{\underbrace{3}_{n-m}} = -\frac{2}{3}$$



① Locating open loop poles and zero

② Parts of the real line which belong to root locus

③ Asymptots

④ Break away points

$$K\,G(s)H(s) = -1$$

$$K = \frac{-1}{G(s)H(s)} \qquad \frac{dK}{ds} = 0 \Rightarrow$$ break away points are calculated

⑤ Angle of departure

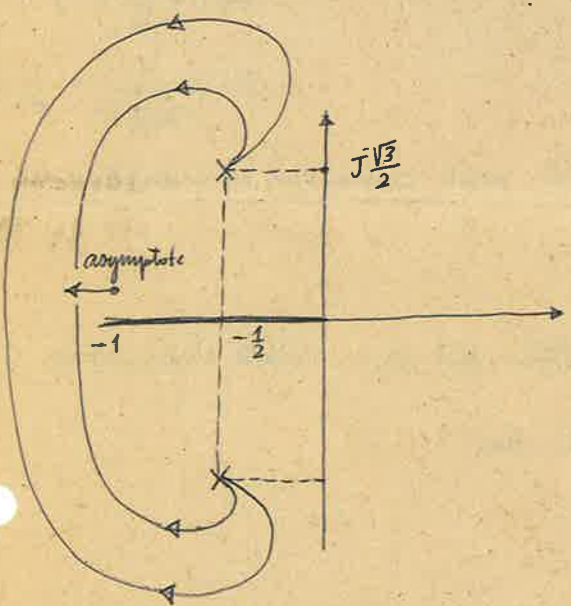
⑥ Calculating the intersections with imaginary axis by using Routh-Hurwitz criteria

Example:

$$G(s)H(s) = \frac{1+Ks}{s(s+1)} = -1$$

$$1 + Ks = -s(s+1) \qquad \frac{1+s^2+s+Ks}{1+s^2+s} = 0$$

$$\frac{Ks}{s^2+s+1} + 1 = 0 \qquad \frac{Ks}{s^2+s+1} = -1 \qquad s_{12} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$



$$\left.\begin{matrix} n=2 \\ m=1 \end{matrix}\right\} \Rightarrow 1 \text{ asymptote} \; : \; K\angle 180° + Z_0$$

$$Z_0 = \frac{-1-(0)}{1} = -1$$

$$s^2+s+1+Ks = 0 \qquad \frac{Ks}{s^2+s+1} = -1$$

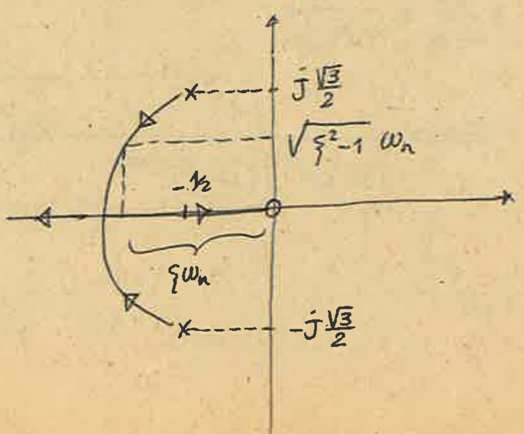
s^2	1	1
s	$K+1$	
1	1	

∴ no intersection with imaginary axis

$$\angle G(s_0)H(s_0) = (2k+1)180°$$

$$\angle s_0 - \angle s_0 - s - \angle s-2$$

$$120° - \theta - 90° = 180° \Rightarrow \theta = 150°$$



$$-K = \frac{s^2+s+1}{s}$$

$$-\frac{dK}{ds} = \frac{(2s+1)s-(s^2+s+1)}{s^2} = 0$$

$$2s^2+s-s^2-s-1 = 0$$

$$s^2 = 1$$

$$s = \begin{matrix} 1 \\ -1 \end{matrix}$$

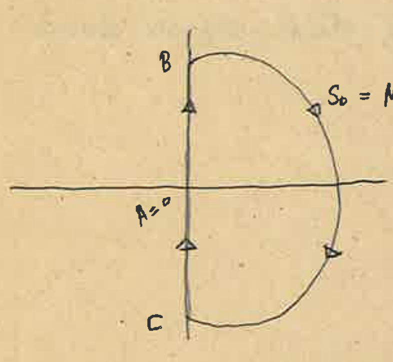
Nyquist Stability Theorem :

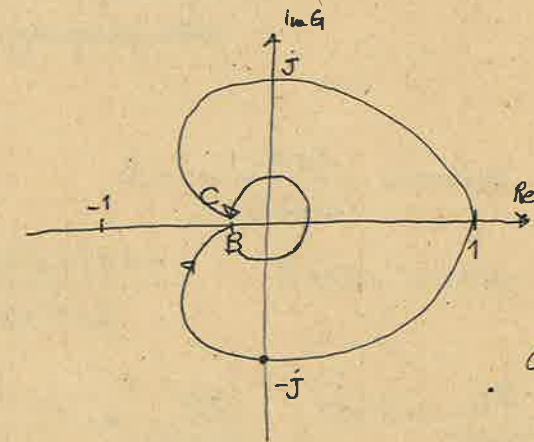
If the open loop transfer function $G(s)H(s)$ has K poles on the right half s plane, then for stability of the closed loop system, the Nyquist plot of $G(s)H(s)$ must encircle $(-1, j0)$ point k times in the counterclockwise direction.

Example :

open loop tr. f.

$$G(s) = \frac{1}{s^2+s+1}$$





$$G(j\omega) = \frac{1}{(1-\omega^2)+j\omega}$$

$$G(j1) = \frac{1}{j} = -j$$

$$G(s) \cong \frac{1}{(M\angle\theta)^2} = \frac{1}{M^2}\angle -2\theta$$

# of poles in RHP = 0
# of encirclements of $(-1, j0)$ = 0 } STABLE

Proof of Nyquist Stability Theorem :

(1) Cauchy Goursat theorem : If a function f is analytic at all points within and on a simple closed contour C, then

$$\oint_C f(s)\,ds = 0$$

(2) Cauchy integral formula : Let f be analytic everywhere within and on a simple closed contour C, taken in clockwise direction

If z_0 is any point interior to C; then

$$f(z_0) = \frac{-1}{2\pi i}\oint_C \frac{f(s)}{s-z_0}\,ds$$

$$F(s) = \frac{P(s)}{Q(s)} = \frac{(s-z_1)^{k_1}(s-z_2)^{k_2}\cdots(s-z_\ell)^{k_\ell}}{(s-p_1)^{m_1}(s-p_2)^{m_2}\cdots(s-p_s)^{m_s}}X(s)$$

$X(s)$ is analytic within and C. The function $F(s)$ has no zeros and poles on C.

$F'(s) = ?$ $\quad F(s) = (s-z_1)^{k_1}F_0(s)$

$$F'(s) = k_1(s-z_1)^{k_1-1}F_0(s) + (s-z_1)^{k_1}F_0'(s)$$

$$\frac{F'(s)}{F(s)} = \frac{k_1\cancel{(s-z_1)^{k_1-1}}\cancel{F_0(s)}}{\cancel{(s-z_1)^{k}F_0(s)}} + \frac{\cancel{(s-z_1)^{k_1}}F_0'(s)}{\cancel{(s-z_1)^{k_1}}F_0(s)} = \frac{k_1}{s-z_1} + \frac{F_0'(s)}{F_0(s)}$$

$$\frac{F'(s)}{F(s)} = \frac{k_1}{s-z_1} + \frac{k_2}{s-z_2} + \cdots \frac{k_\ell}{s-z_\ell} + \frac{F_1'(s)}{F_1(s)}, \quad F_1(s) = \boxed{\frac{1}{(s-p_1)^{m_1}\cdots(s-p_\sigma)^{m_\sigma}}X(s)}$$

$$\rightarrow F_1'(s) = \frac{F_2'(s)}{(s-p_1)^{m_1}}$$

it will be continued,

$F(s)$: rational function

C : a simple closed curve in s-domain

Clockwise direction

Γ : a closed curve (not simple)

such that $F(s) : C \rightarrow \Gamma$

$$\oint_C \frac{F'(s)}{F(s)} ds = -2\pi i (Z - P)$$

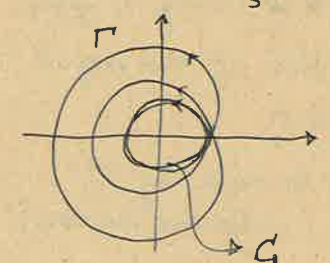
Z : # of zeros of $F(s)$ inside C

P : " " poles " " " "

$$\oint_C \frac{F'(s)}{F(s)} ds = \oint_\Gamma \frac{1}{z} dz \quad (z \triangleq F(s))$$

$$= -2\pi i N$$

# of encirclements of the origin by the curve Γ



$$\oint_\Gamma \frac{1}{z} dz = -\cancel{2\pi i} + \cancel{2\pi i} - 2\pi i$$

$$\boxed{(Z - P) = N}$$

Examples :

Example I :

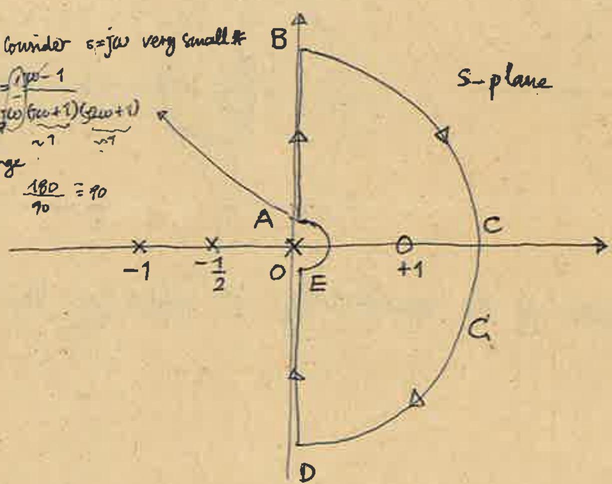
$$F(s) = \frac{s-1}{s(s+1)(2s+1)}$$

Consider $s = j\omega$ very small #

$F(s) = \frac{j\omega - 1}{j\omega \underbrace{(j\omega+1)}_{\sim 1}\underbrace{(2j\omega+1)}_{\sim 1}}$

$|F(s)|$ very large

angle $\cdot$ $\frac{180}{90} \cong 90$



$F(s)$

at $\underline{\underline{A}}$: $\frac{1}{\omega} \angle 90°$

at $\underline{B}$: $s = j\omega$
ω is very large.

$F(j\omega) \cong \frac{1}{2(j\omega)^2} = \frac{1}{2\omega^2} \angle 180$

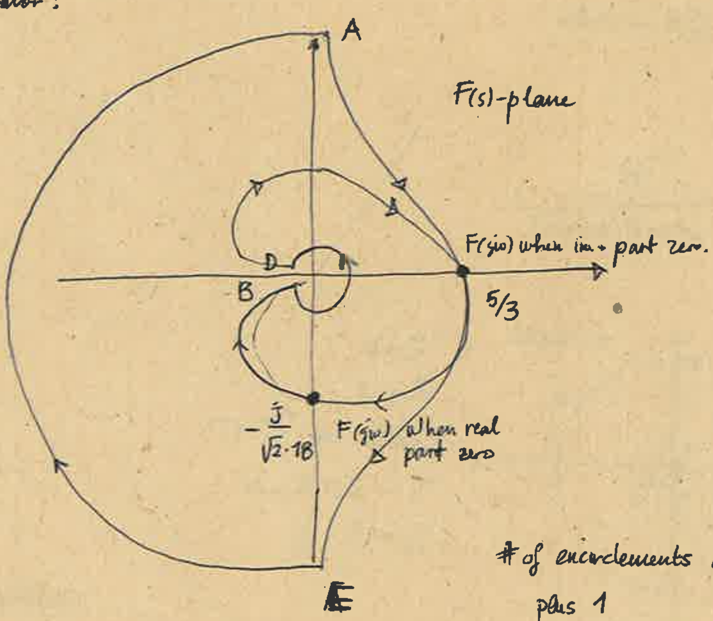
between A and B

$s = j\omega$

$$F(j\omega) = \frac{j\omega - 1}{j\omega(j\omega+1)(2j\omega+1)}$$

multiply by complex conjugate of denominator :

$$F(j\omega) = \frac{4\omega - 2\omega^3 + j(5\omega^2+1)}{\omega[9\omega^2 + (1-2\omega^2)^2]}$$



between B and D

$S_0 = M \angle \theta$ "take very large $|S_0|$"

$$F(S_0) \cong \frac{1}{2S_0^2} = \frac{1}{2M^2 \angle 2\theta} = \frac{1}{2M^2} \angle -2\theta$$

between D and E

– symmetric –

between E and A

$S_0 \cong \epsilon \angle \theta$

$$F(S_0) \cong \frac{-1}{\epsilon \angle \theta} = \frac{1}{\epsilon} \angle 180 - \theta$$

# of encirclements of origin :

plus 1

because encirclement is in clockwise direction

$$\left.\begin{array}{l} N = +1 \\ Z = 1 \\ P = 0 \end{array}\right\} N = Z - P$$

Open loop transfer function: $G(s)H(s) = \dfrac{P(s)}{Q(s)}$ $\longrightarrow Z_0$: # of zeros of $P(s)$; $\longrightarrow P_0$: # of zeros of $Q(s)$

C on s-domain $G(s)H(s) : C \rightarrow \Gamma$

Characteristic eq: $1 + G(s)H(s) = 1 + \dfrac{P(s)}{Q(s)} = \dfrac{Q(s)+P(s)}{Q(s)}$ $\longrightarrow Z_1$: # of zeros of $Q(s)+P(s)$ (in C); $\longrightarrow P_1$: # " " " $Q(s)$ (" ")

$P_1 = P_0$

$P_1 = P_0$

P_c : # of poles of the closed loop system in C,
(RHP poles)
we want them out ~~to be~~ for stability of system.

Closed loop system is stable if and only if $P_c = 0$

Set $1 + H(s)G(s) = C \rightarrow \Gamma_1$

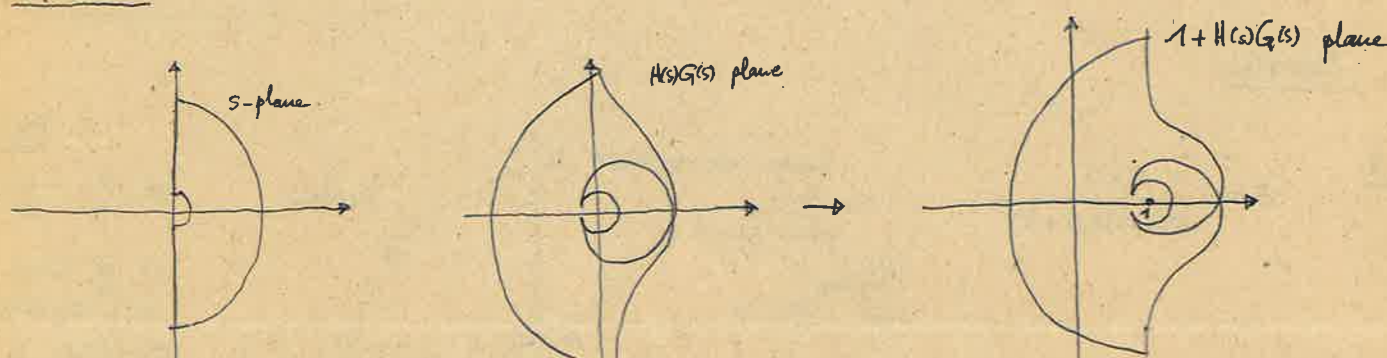
Set N_1 # of encirclements of the ~~origin~~ by Γ_1
(in clockwise direction)

# of closed loop poles; # of open loop poles

$N_{-1} = Z_1 - P_1 = P_c - P_0$

$N_{-1} = -P_0$

imp. note:

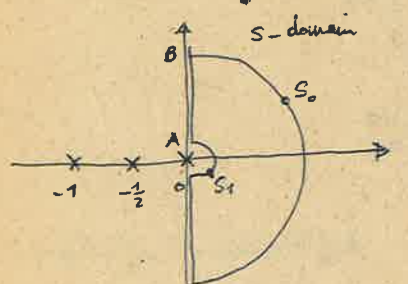


N_{-1} : encirclements of the point (-1) by Γ which is the map of C by $G(s)H(s)$

$N_{-1} = 1$
$N_{-1} = P_c - P_0$
$+1 = P_c - 0$
$P_c = 1 \Leftarrow$ unstable.

Example 2

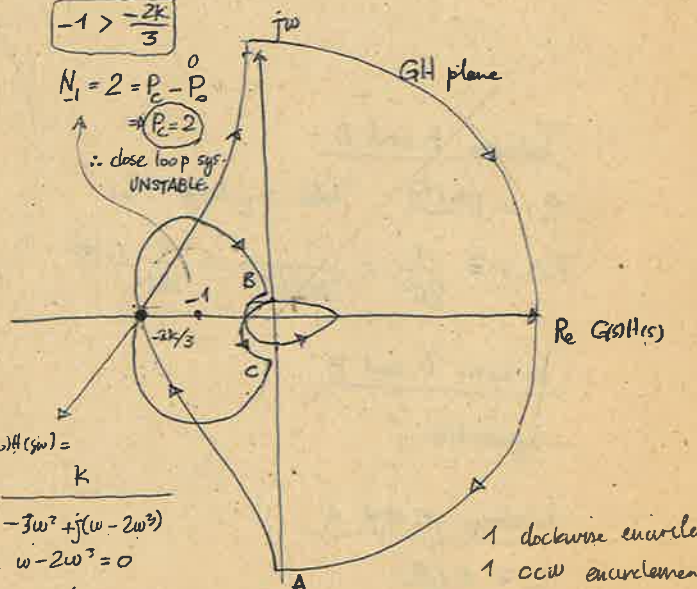
$$G(s)H(s) = \frac{K}{s(s+1)(2s+1)}$$



$P_0 = 0$
$Z_0 = 0$
$G(s)H(s) \Rightarrow C \rightarrow \Gamma$
$N_0 = Z_0 - P_0 = 0$
$N_{-1} = P_c - P_0$

WHEN $\boxed{-1 > \frac{-2k}{3}}$

$N_1 = 2 = P_c - P_0$, $P_0 = 0$
$\Rightarrow P_c = 2$
∴ close loop sys. UNSTABLE



$G(j\omega)H(j\omega) = \dfrac{K}{-3\omega^2 + j(\omega - 2\omega^3)}$

∴ $\omega - 2\omega^3 = 0$

$\omega = \frac{1}{\sqrt{2}}$

$G(\frac{1}{\sqrt{2}})H(\frac{1}{\sqrt{2}}) = \boxed{\frac{-2k}{3}}$

1 clockwise encirclement
1 ccw encirclement
TOTAL ENCIRC : 0

$N_0 = 0 = Z_0 - P_0$; $Z_0 = 0$, $P_0 = 0$

at A : $s = j\omega$ very small

$$G(j\omega)H(j\omega) \cong \frac{K}{j\omega} = \frac{K}{\omega}\angle -90^\circ$$

at B : $s = j\omega$ ω very large

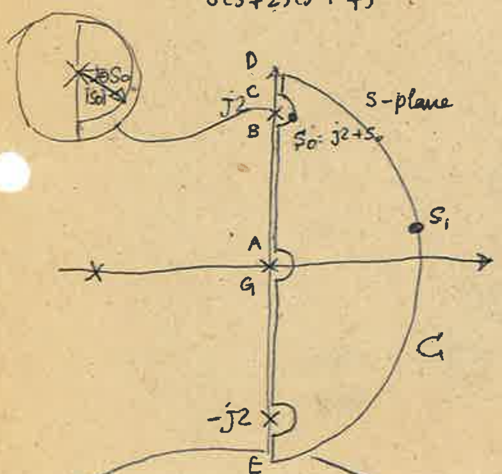
$$G(j\omega)H(j\omega) \cong \frac{K}{2(j\omega)^3} = \frac{K}{2\omega^3}\angle 90^\circ$$

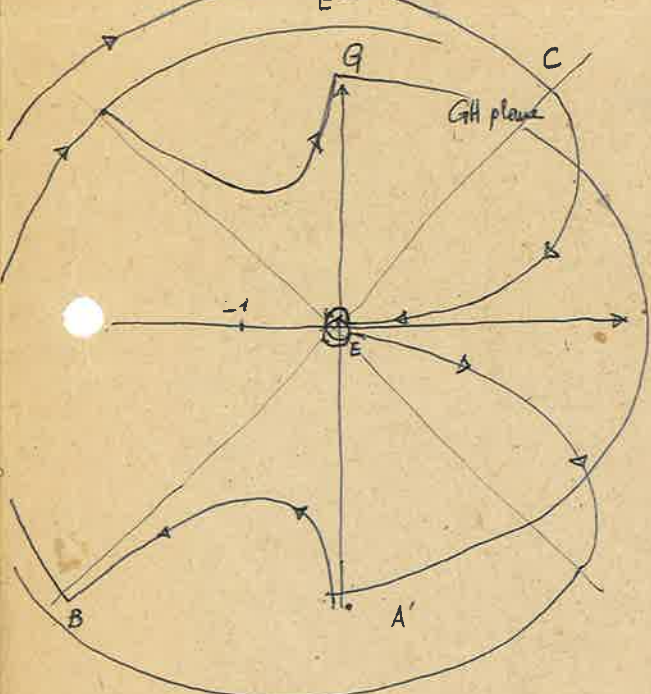
$$G(s_o)H(s_o) \cong \frac{K}{2s_o^3} = \frac{K}{2M^3}\angle -3\theta^\circ$$

$$G(s_1)H(s_1) \cong \frac{K}{s_1} = \frac{K}{\varepsilon}\angle -\theta$$

Example 3

$$G(s)H(s) = \frac{50}{s(s+2)(s^2+4)}$$





at point A $s = j\omega$ ω: very small

$$G(j\omega)H(j\omega) = \frac{50}{8j\omega} = \frac{50}{8\omega}\angle -90^\circ$$

at point B

$$s = j(2-\varepsilon) \cong j2$$

$$G(j\omega)H(j\omega) \cong \frac{50}{j2(2+j2)X}$$

$$X: \left[[j(2-\varepsilon)]^2+4\right] = $$

$$\cong \frac{\cancel{50}\,25}{\angle 90^\circ \sqrt{8}\angle 45^\circ\ 4\varepsilon}$$

$$X \cong 4\varepsilon$$

$$\cong M\angle -135^\circ$$

very large

$$G(s_o)H(s_o) \cong \frac{50}{j2(j2+2)Y}$$

$$Y = (j2+s_o)^2+4 \cong -4js_o \cong 4|s_o|\angle 90^\circ+\theta$$

$$\cong \frac{50}{2\sqrt{8}\angle 90^\circ+45^\circ+90^\circ+\theta \cdot 4|s_o|}$$

$$\cong M\angle -225-\theta$$

very large

at point D

$$G(j\omega)H(j\omega) \cong \frac{50}{j\omega(j\omega+2)(-\omega^2+4)}$$ ω: very large

$$= \frac{50}{-\omega^2+4}\ \frac{1}{2\omega j-\omega_2}$$

no intersection with $j\omega$ and σ axes.

on large circle :

$$G(s_1)H(s_1) \cong \frac{50}{s_1^4} = \frac{50}{|s_1|^4}\angle \theta 4$$

# of encirclements of origin 0 $N_o = 0$ $Z_o = 0$ $P_o = 0$

# of encirclements of -1 : $N_c = 2$ UNSTABLE closed loop system 2 poles at RHP

23680

Bode Plot

$20\log|G(j\omega)|$ versus $\log\omega$

$\angle G(j\omega)$ versus $\log\omega$

Example

$G(s) = as+1 \qquad a>0$

a) $20\log|G(j\omega)| = 20\log|aj\omega+1| = 20\log\sqrt{a^2\omega^2+1}$

i) $\omega \ll \frac{1}{a} \qquad a^2\omega^2 \ll 1$

$\sqrt{a^2\omega^2+1} \cong 1 \Rightarrow 20\log\sqrt{a^2\omega^2+1} \cong 0$

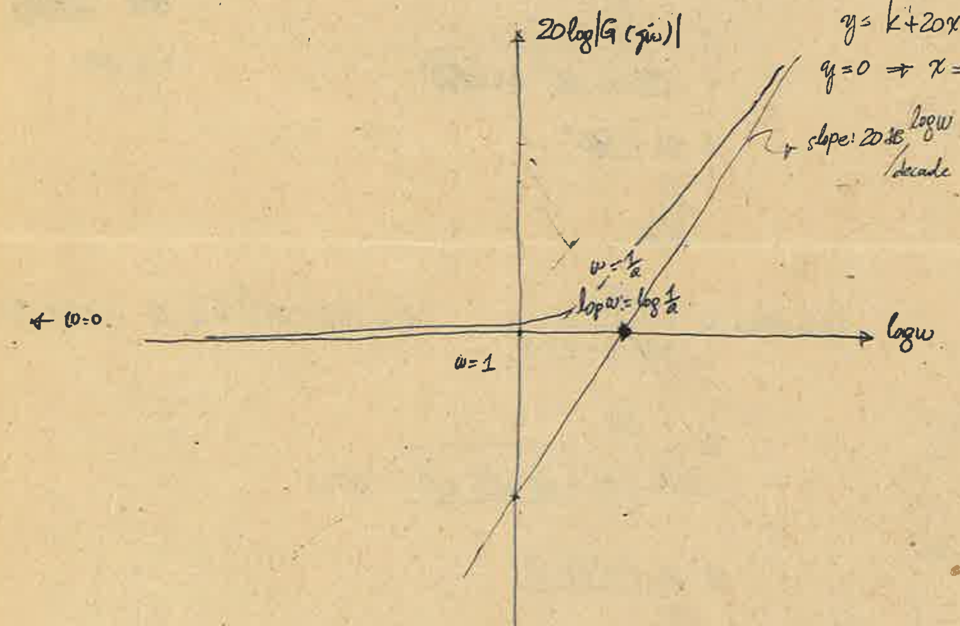
ii) $\omega \gg \frac{1}{a} \qquad a^2\omega^2 \gg 1$

$$20\log|ja\omega+1| \cong 20\log\sqrt{a^2\omega^2} = 20\log a\omega = \underbrace{20\log a}_{k} + \underbrace{20\log\omega}_{20x}$$

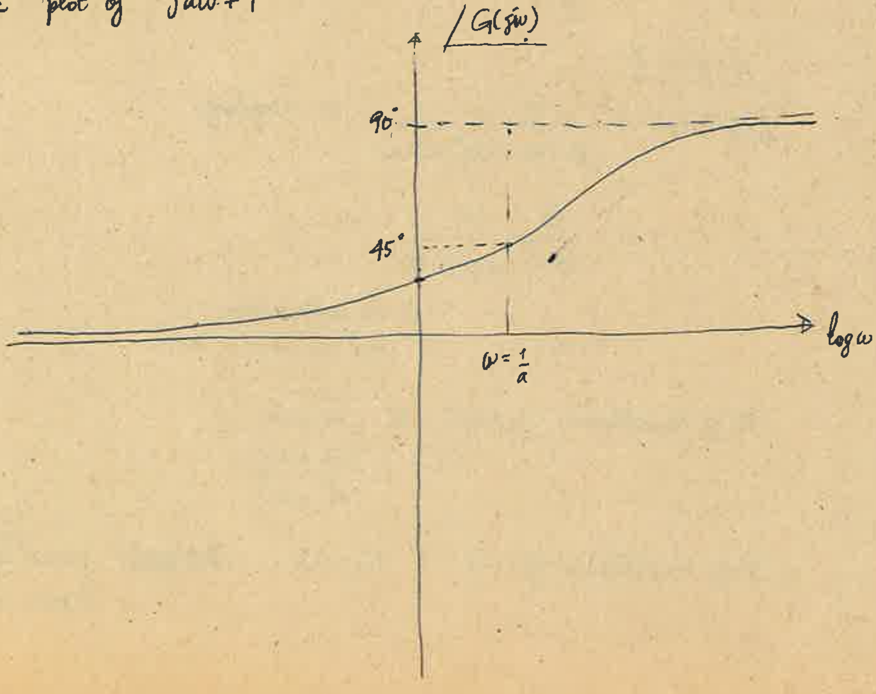
$y = k+20x$

$y=0 \Rightarrow x = \frac{-k}{20}$

$\log\omega = \frac{-20\log a}{20} = -\log a = \log\frac{1}{a}$



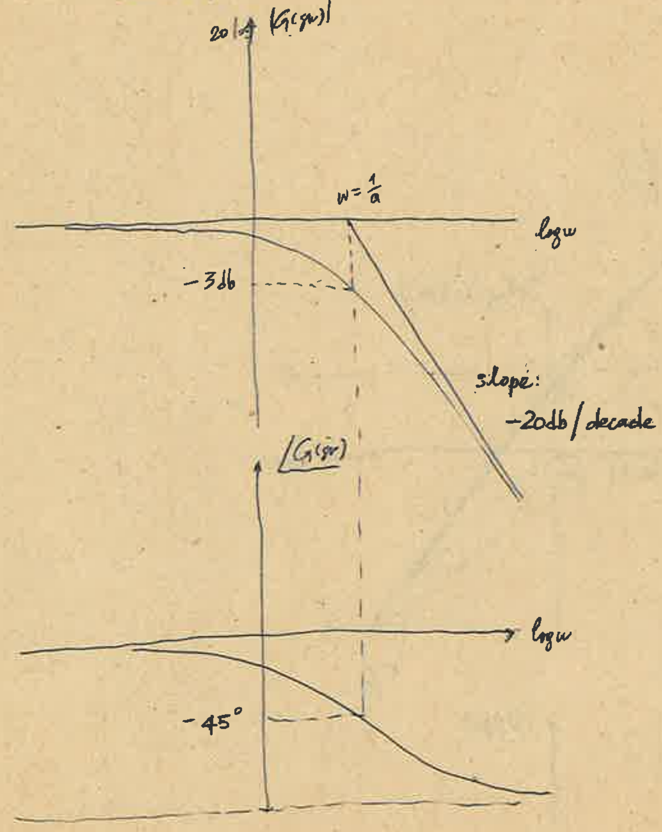
b) Phase plot of $ja\omega+1$



Example

$$G(s) = \frac{1}{as + 1}$$

$$20\log|G(j\omega)| = 20\log\left|\frac{1}{ja\omega+1}\right| = -20\log|ja\omega+1|$$



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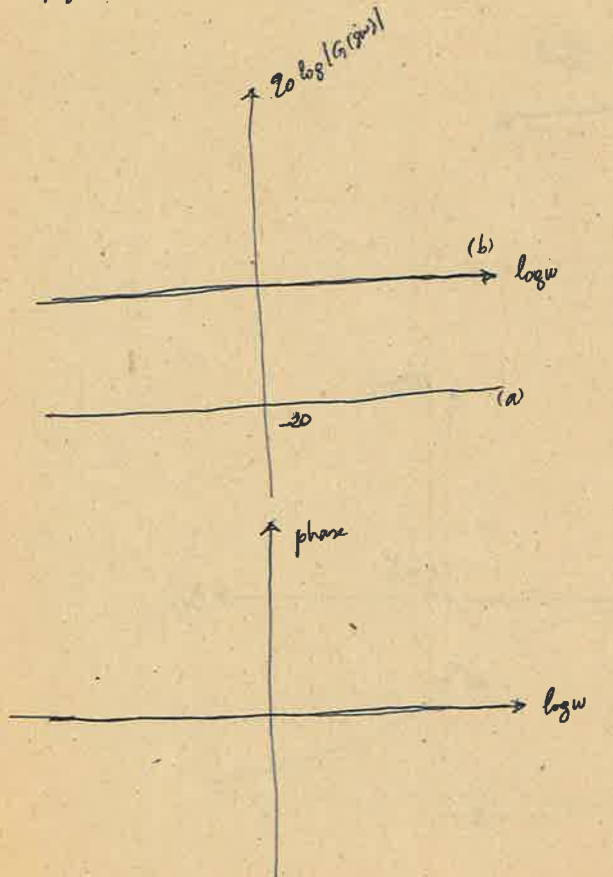
Example :

$G(s) = K \quad K>0$

$\phi(j\omega) = 0°$

$|G(j\omega)| = K \Rightarrow 20\log K$



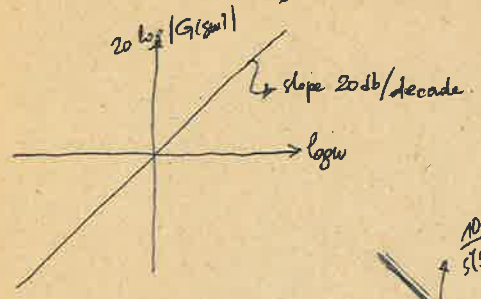
(a) $K = \frac{1}{10} \Rightarrow 20\log\frac{1}{10} = -20$

(b) $K = 1 \Rightarrow 0$

Example

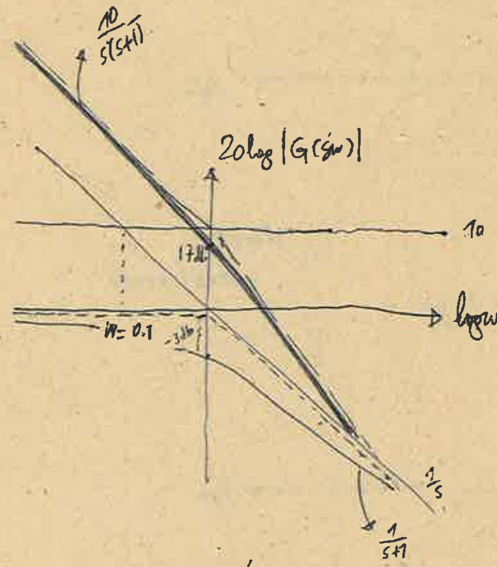
$G(s) = s$

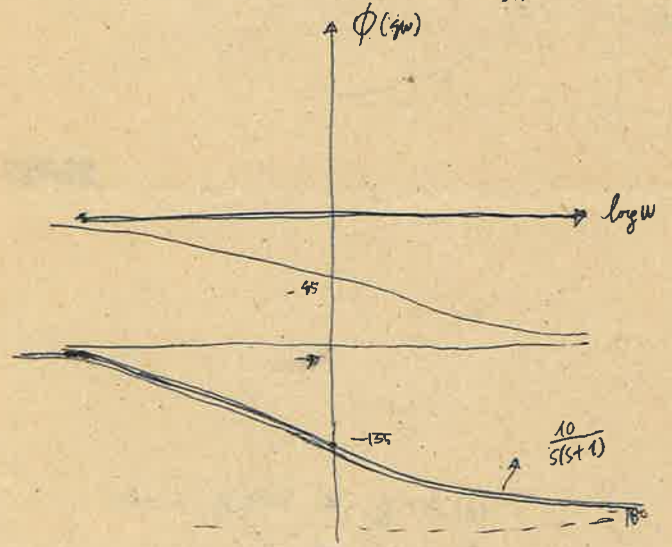
$20 \log |G(j\omega)| = 20 \underbrace{\log \omega}_{x} = 20x \qquad \phi(j\omega) = 90^\circ$



Example

$G(s) = \dfrac{10}{s(s+1)}$

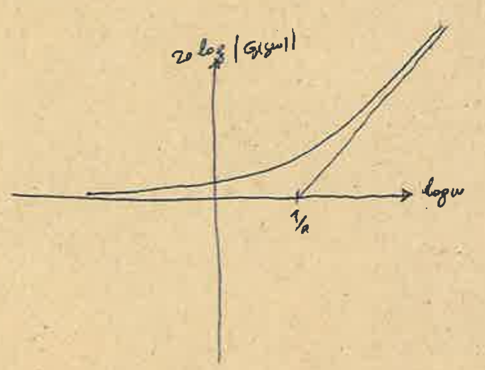


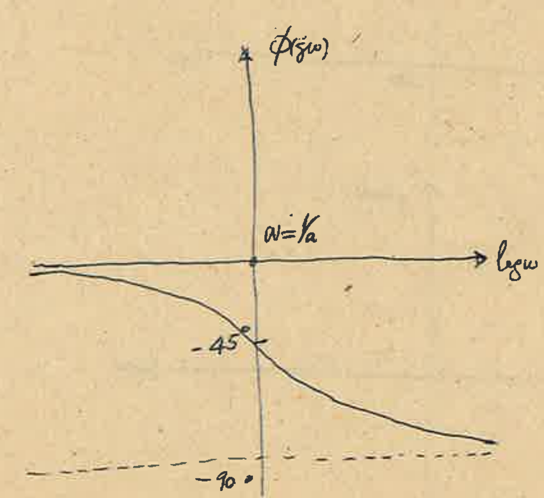


Example

$G(s) = -as + 1, \quad a > 0$

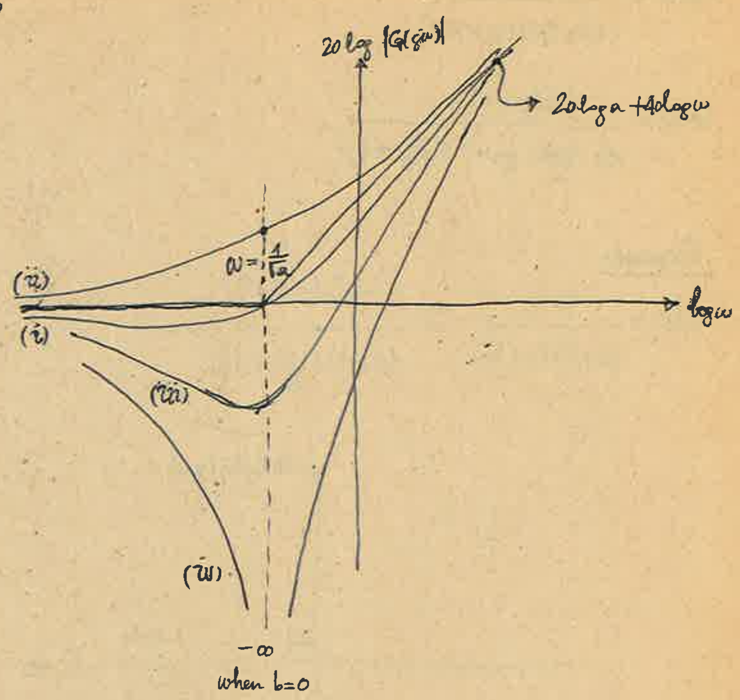
$|G(j\omega)| = |-ja\omega + 1| = |ja\omega + 1|$





Example :

$G(s) = as^2 + bs + 1$ it cannot be written as the multiplication of two first order term $a, b > 0$



$|G(j\omega)| = |1 - a\omega^2 + jb\omega| = \sqrt{(1-a\omega^2)^2 + b^2\omega^2}$

$20 \log |G(j\omega)| = 20 \log \sqrt{(1-a\omega^2)^2 + b^2\omega^2}$

i) $a\omega^2 \ll 1 \Rightarrow \omega \ll \frac{1}{\sqrt{a}}$

then the approximate value of $20\log|G(s)| \cong 20 \log \sqrt{1} = 0$

ii) $\omega \gg \frac{1}{\sqrt{a}}$

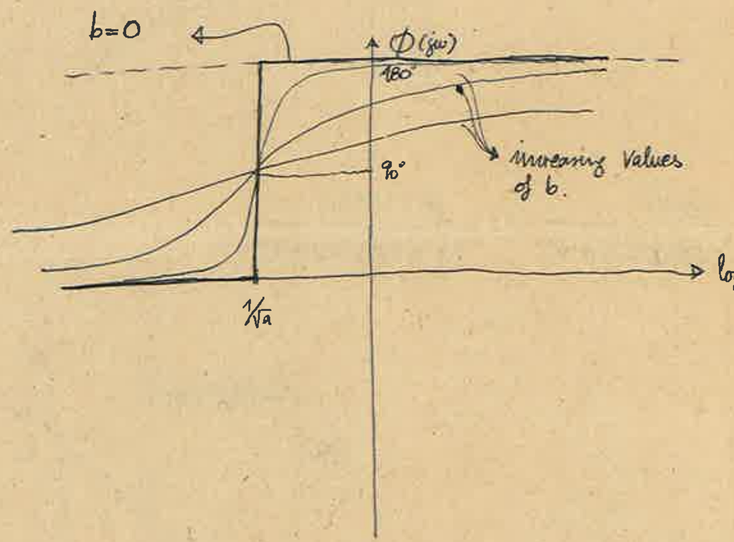
$20 \log |G(j\omega)| \cong 20 \log \sqrt{(a\omega^2)^2} = 20 \log a\omega^2$

$= \underbrace{20 \log a}_{\text{constant}} + 40 \underbrace{\log \omega}_{x}$

at $\omega = \frac{1}{\sqrt{a}}$ $\left|G\left(j\frac{1}{\sqrt{a}}\right)\right| = \sqrt{\frac{b^2}{a}} = \frac{b}{\sqrt{a}} \Rightarrow 20 \log \frac{b}{\sqrt{a}}$

for different values of a and b :

(i) $b = \sqrt{a}$
(ii) $b > \sqrt{a}$
(iii) $b < \sqrt{a}$
(iv) $b = 0$



Phase of $as^2 + bs + 1$ $a > 0, b > 0$

$G(j\omega) = (1 - a\omega^2) + jb\omega$

a) $\omega \ll \frac{1}{\sqrt{a}} \Rightarrow \phi(j\omega) \cong 0$

$\omega = \frac{1}{\sqrt{a}} \Rightarrow \phi(j\omega) \cong 90°$

$\omega \gg \frac{1}{\sqrt{a}} \Rightarrow \phi(j\omega) \cong 180°$

Example :

$G(s) = -as^2 + bs + 1$ $a, b > 0$
the roots are real, we can factor it into two first order polynomial

$G(s) = (-s + s_1)(s - s_2)$

Example

$G(s) = -as^2 - bs + 1$ $a, b > 0$

$G(j\omega) = (1 - a\omega^2) - jb\omega$ ← magnitude will be same with that one at the top of the page.

$\phi(j\omega)$ is equal to $-1 \times$ of the one at the top of the page.

Example:

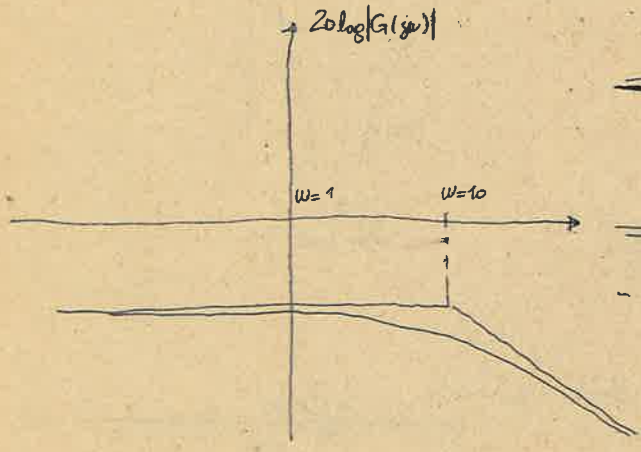
$$G(s) = \frac{100(s+1)}{(s+10)(s+100)}$$

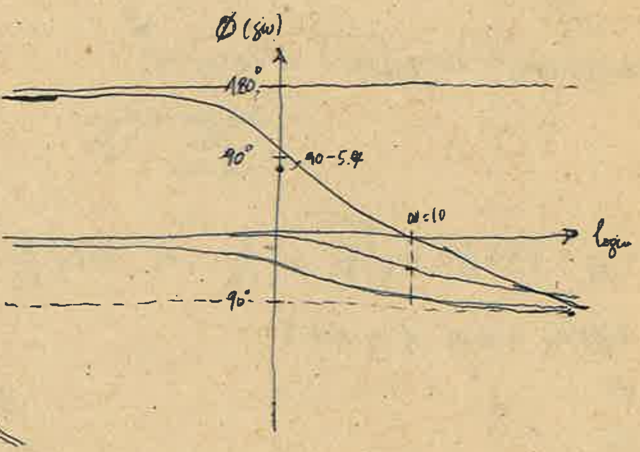
$$G(s) = \frac{\cancel{100}(s+1)}{10\cdot\cancel{100}\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}$$

Example:

$$G(s) = \frac{s-1}{(s+1)(s+10)} = \frac{-\frac{1}{10}(-s+1)}{(s+1)\left(\frac{s}{10}+1\right)}$$

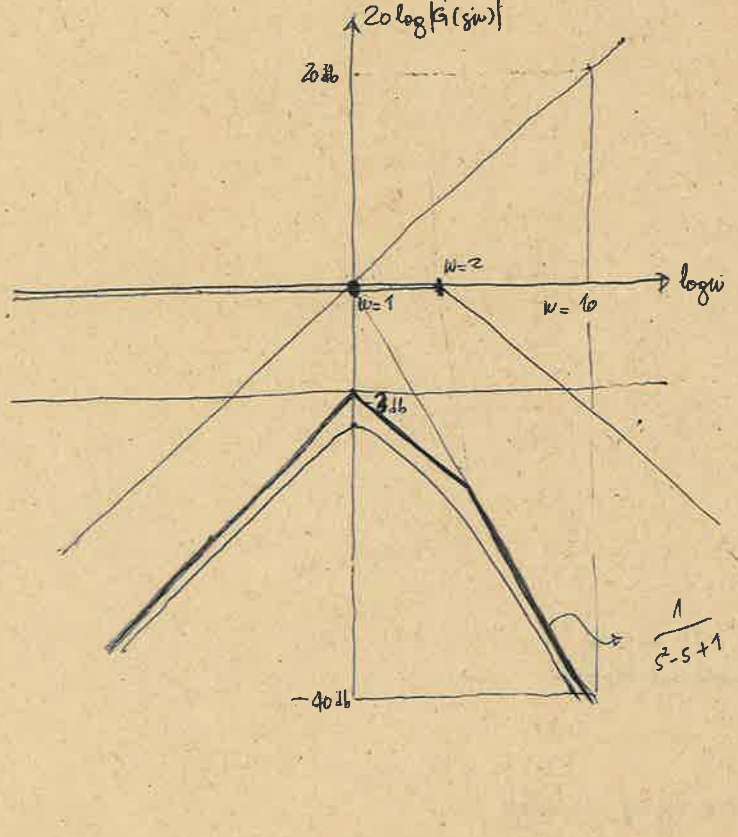
$|-s+1| = |s+1|$ so they cancel eachother.





Example:

$$G(s) = \frac{s(s+1)}{(s+2)(s^2-s+1)} = \frac{\frac{1}{2}s(s+1)}{\left(\frac{s}{2}+1\right)(s^2-s+1)}$$

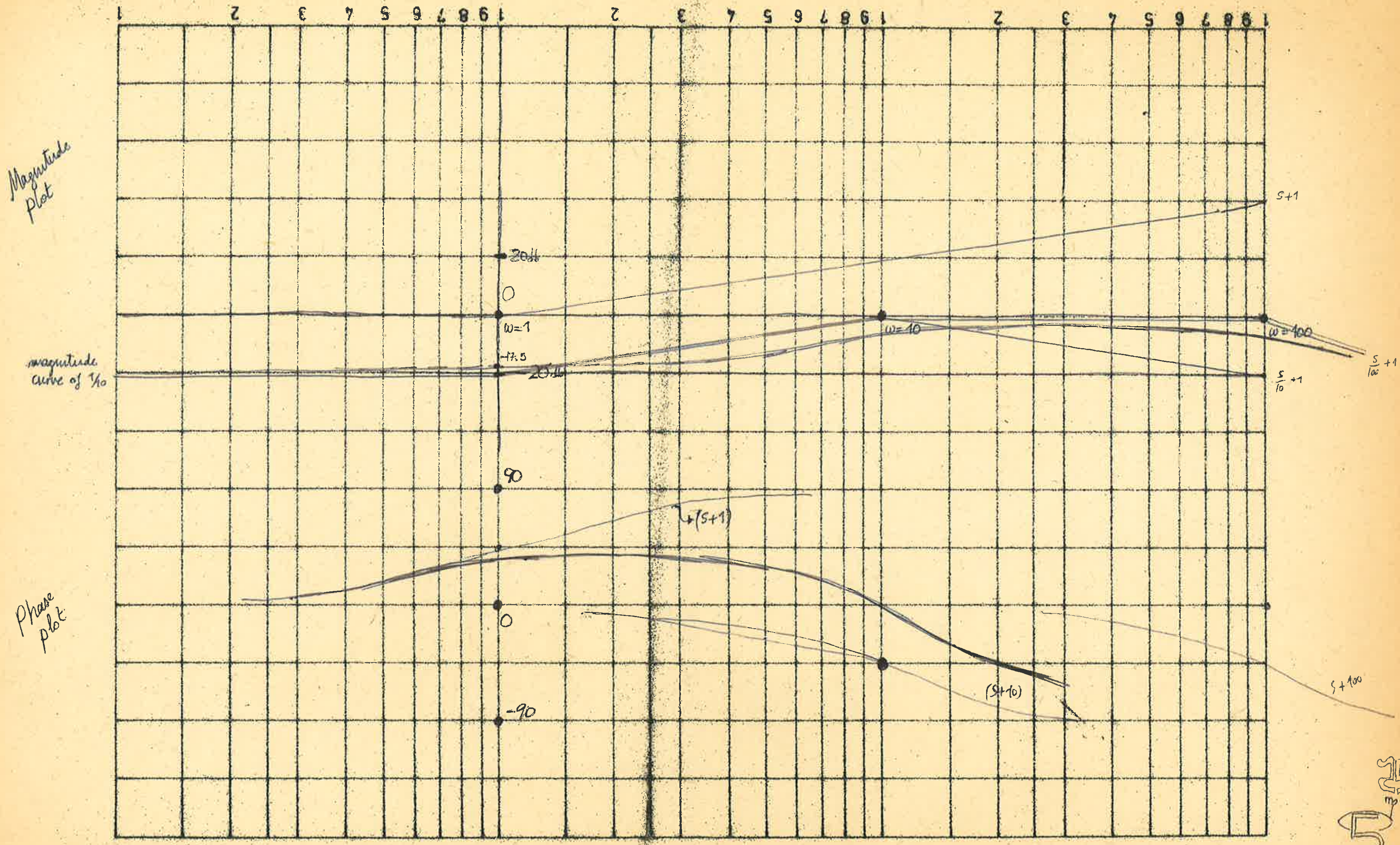


at $w=1$ $\quad 20\log|G(j\omega)| = -6+0+3-20\log\left|j\frac{1}{2}+1\right|$

$$\underline{-20\log 1}$$

$$20\log|G(j\omega)| = -3-\overset{0}{20\log}\left|\frac{\sqrt{5}}{2}\right|$$

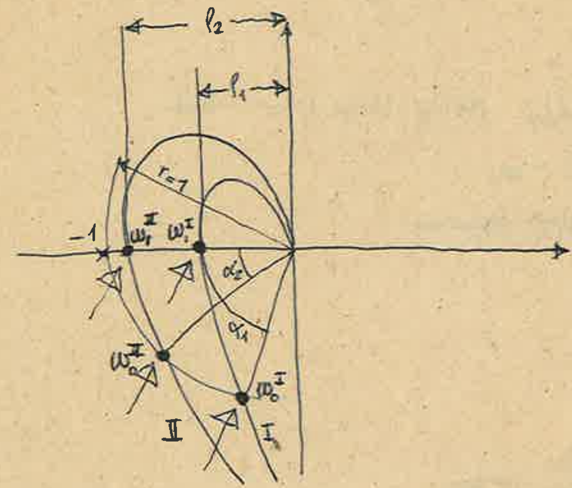
$$G(s) = \frac{100(s+1)}{(s+10)(s+100)}$$



Relative stability Analysis

Assumption: The system that we shall study will be "unity feedback" and "minimum phase".

Definition: If the open loop transfer function $G(s)$ of a system has no rhp poles and zeros then this system called minimum phase.



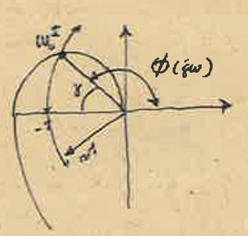
Definition: Gain cross-over frequency: w_o is the frequency where $|G(jw_o)| = 1$

Definition: Phase crossover frequency: w_1 is the frequency where $\phi(jw_1) = \angle G(jw_1) = -180^\circ$

Definition: Phase margin: is the angle γ where

$$\gamma = 180 + \phi(jw_o)$$

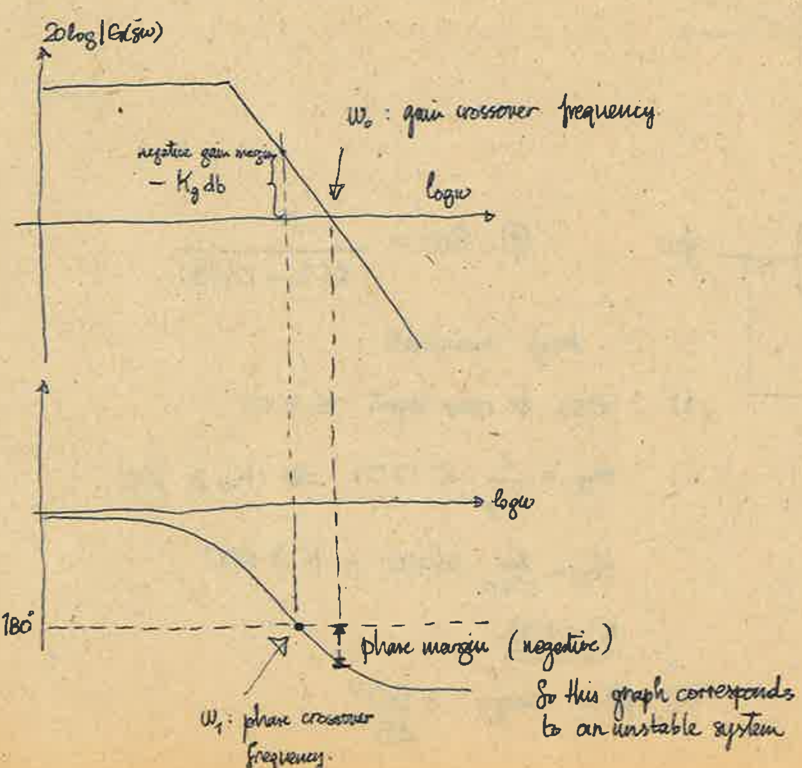
unstable system:



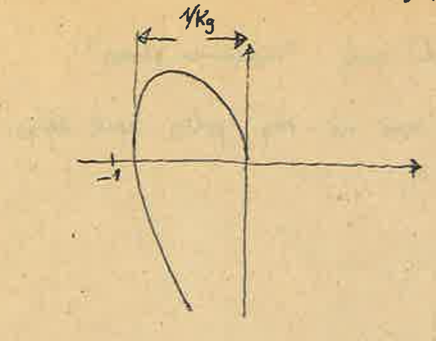
a) For an unstable system phase margin defined above is negative.

b) For a stable system, phase margin is positive.

Bode Plot:



Definition : Gain Margin : $K_g = \dfrac{1}{|G(j\omega_1)|}$ where ω_1 : phase crossover frequency.

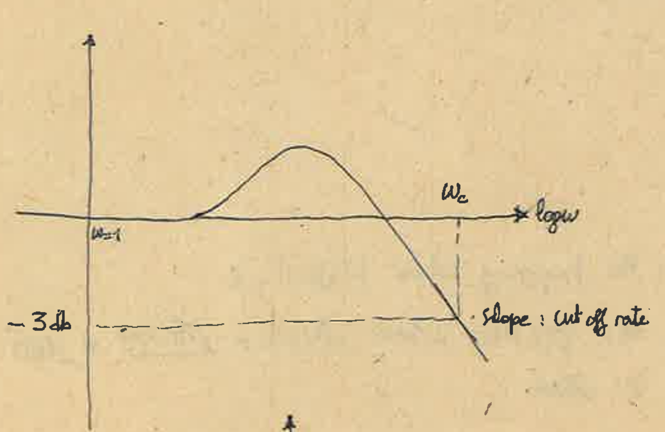


$$K_g\,db = 20\log\frac{1}{|G(j\omega)|}$$
$$= -20\log|G(j\omega)|$$

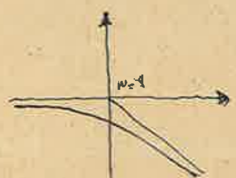
Definition : Cutoff frequency : ω_c is the frequency where $20\log|G(j\omega_c)| = -3\,db$

Definition : Bandwidth : the length of interval $[0,\omega_c] = \omega_c$

Definition : Cut off rate : slope of $|G(j\omega)|$ at the cut off frequency.

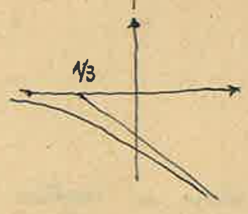


Example : I : $\dfrac{1}{s+1}$



I// $\omega_c = 1$ $BW = 1$ cut off rate :

II $\dfrac{1}{3s+1}$

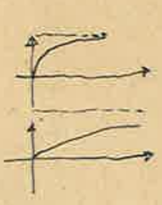


II/ $\omega_c = \frac{1}{3}$ $BW = \frac{1}{3}$

response to unit step :

$y_I(t) = -e^{-t} + 1$

$y_{II}(t) = -\frac{1}{3}e^{-\frac{1}{3}} + \frac{1}{3}$



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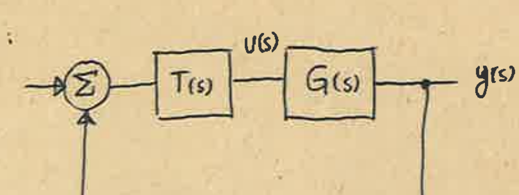
Example :



① $G(s) = \dfrac{K}{s(1+0.4s)}$

design requirements :

i) ess to ramp input ≤ 0.01

$e_{ss} = \dfrac{1}{K_v} \leq 0.01 \Rightarrow K_v \geq 100$

$K_v = \lim_{s\to 0} sG(s) = K \geq 100$

$\underline{K = 100}$

ii) phase margin $\geq$ ~~50°~~ 45°

Phase – lead compensation

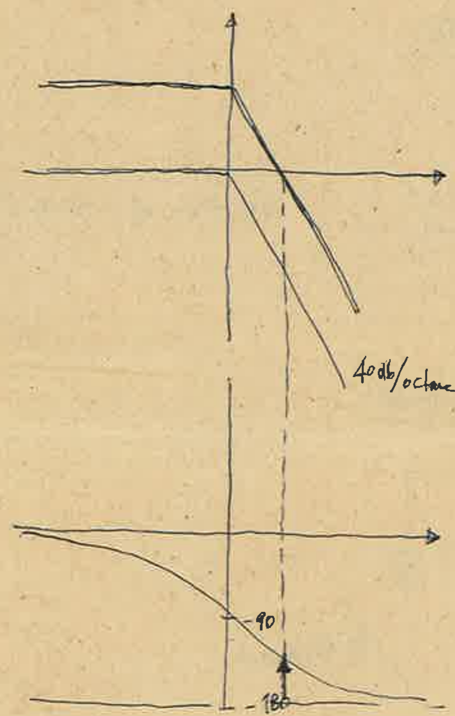
Limitations of phase - lead compensation:

(1) For unstable systems, the additional phase lead necessary to obtain a certain specified phase margin is large. This requires a large value for $\frac{1}{\alpha}$ $\left(\sin \varphi_m = \frac{1-\alpha}{1+\alpha}\right)$
In practice, the value of $\frac{1}{\alpha}$ is not chosen greater than 15.

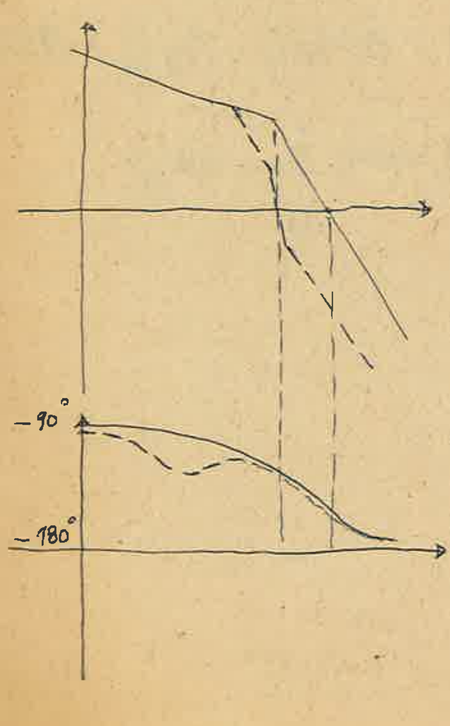
(2) For systems with especially low or negative damping ratios, if phase shift decreases rapidly, near the gain crossover frequency, phase lead at the new gain crossover is added to a much smaller phase angle than that at the old gain crossover. the desired phase margin may be realised only by using a very large value of $\frac{1}{\alpha}$.

Example:

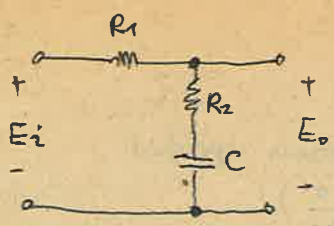
$$G(s) = \frac{K}{(s+1)^2}$$



Lag compensator:



Lag network

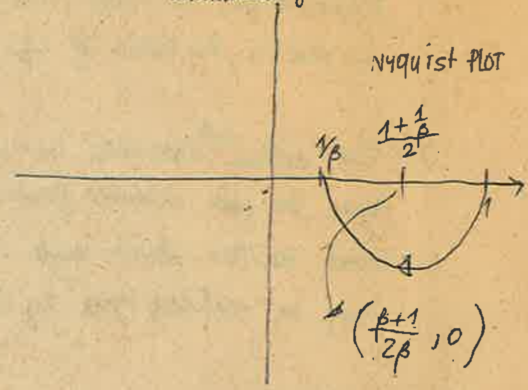


$$\frac{E_o(s)}{E_i(s)} = \frac{R_2+\frac{1}{sC}}{R_1+R_2+\frac{1}{sC}} = \frac{\overbrace{sR_2C}^{T}+1}{\underbrace{sC(R_1+R_2)}_{T\beta}+1} = \frac{sT+1}{sT\beta+1}$$

phase of denominator greater

$T \triangleq R_2C$

$$\underbrace{R_2C}_{T}\underbrace{\frac{(R_1+R_2)}{R_2}}_{\beta}$$



$$T(j\omega) = \frac{j\omega T+1}{j\omega\beta T+1} = \frac{(j\omega T+1)(1-j\omega T\beta)}{\underbrace{(\omega^2T^2\beta^2+1)}_{\Delta}}$$

$$= \underbrace{\frac{1+\omega^2T^2\beta}{\Delta}}_{x} + j\underbrace{\frac{\omega T-\omega T\beta}{\Delta}}_{y}$$

magnitude of Transfer function

$$\left(x-\frac{\beta+1}{2\beta}\right)^2+y^2 = \left(\frac{\beta-1}{2\beta}\right)^2$$

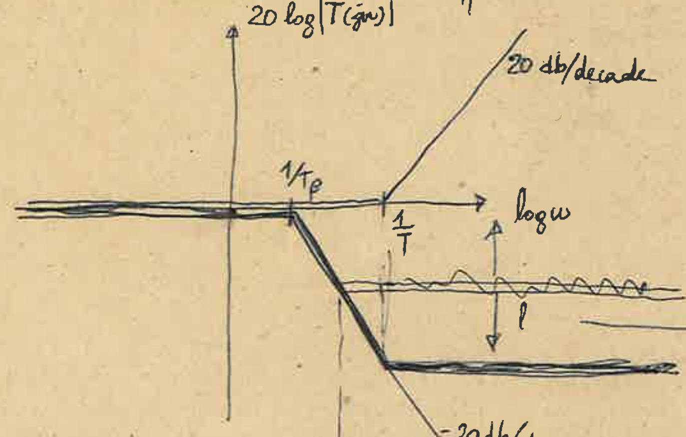
Equation of a semicircle

Bode Plot of the phase lag network

$$T(s) = \frac{Ts+1}{T\beta s+1} \qquad \beta>1$$

corner freq: $\frac{1}{T}$

corner freq: $\frac{1}{T\beta}$



$$l = 20\left(\log\frac{1}{T}-\log\frac{1}{T\beta}\right) = 20\log\beta$$

$$l = 20\log\frac{\frac{1}{T}}{\frac{1}{T\beta}} = 20\log\beta$$