Subjects:

Jordan and Balmain
Plonsey and Collin
Lorrain and Corson

Time varying electric and magnetic fields

1 - Maxwell's equations
2 - Energy relationships in fields and its relations to circuits.
3 - Plane waves.
4 - Reflection and refraction of plane waves.
5 - Transmission lines
6 - General boundary value problems.

Differential equations in Electromagnetic fields.
(Wave phenomena)

$$\nabla\times\vec{B} = \mu\vec{J} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\underbrace{\nabla\times(\nabla\times\vec{B})}_{\text{expand}} = \mu\nabla\times\vec{J} + \mu\varepsilon\frac{\partial}{\partial t}(\nabla\times\vec{E})$$

$$\cancel{\nabla(\underbrace{\nabla\cdot\vec{B}}_{0})} - \nabla^2\vec{B} = \mu\nabla\times\vec{J} - \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\boxed{\nabla^2\vec{B} - \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} = -\mu\nabla\times\vec{J}}$$

$$\nabla\times(\nabla\times\vec{E}) = -\frac{\partial}{\partial t}\nabla\times\vec{B} = -\frac{\partial}{\partial t}\left(\mu\vec{J} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla(\nabla\vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial\vec{J}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$\nabla\vec{E} = \frac{\rho}{\varepsilon}$ known.

$$\boxed{\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \mu\frac{\partial\vec{J}}{\partial t} + \frac{1}{\varepsilon}\nabla\rho}$$

We usually use auxiliary potentials:

$\vec{A}$: Vector potential

Φ : Scalar potential

$$\nabla\vec{B} = 0 \Rightarrow \vec{B} = \nabla\times\vec{A}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\nabla\times\frac{\partial}{\partial t}\vec{A}$$

$$\nabla\times\left(\vec{E} + \frac{\partial A}{\partial t}\right) = 0$$

$$\boxed{\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t}}$$

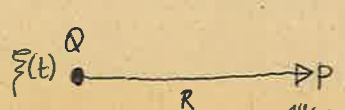
$$\boxed{\nabla\vec{A} + \mu\varepsilon\frac{\partial\Phi}{\partial t} = 0}$$ Lorentz equation

Wave Propagation 14/10/1980

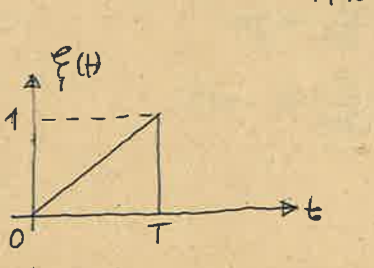
Point source : $\xi(t)$
Location : Q

Huygen's source: Point source; producing spherical waves

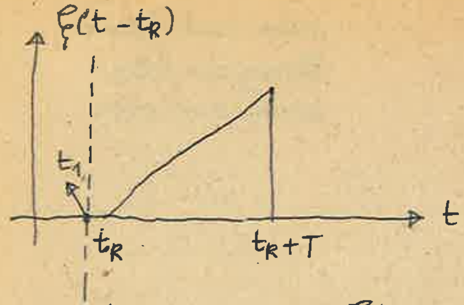
Wave function $\Psi(R,t)$ considered at P

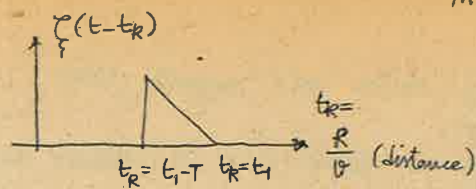


$$\Psi(R,t) = \frac{\xi\left(t - \frac{R}{v}\right)}{4\pi R}$$

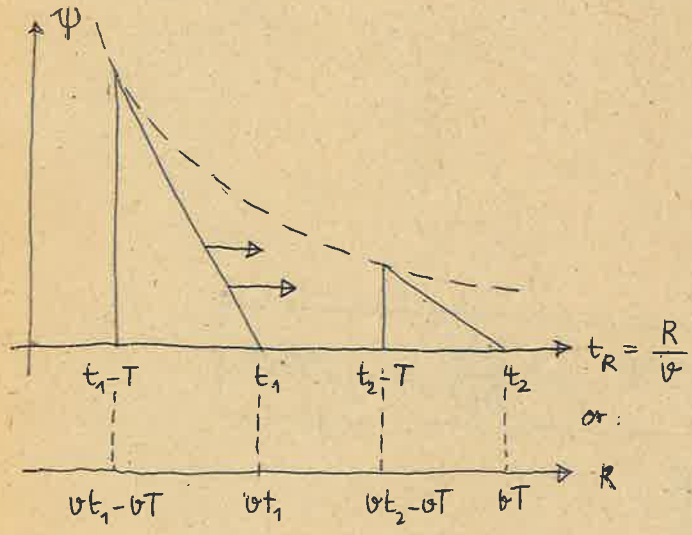


time variation of the source





when $t_R \geqslant t_1$ $\quad \xi(t_1 - t_R) = 0$



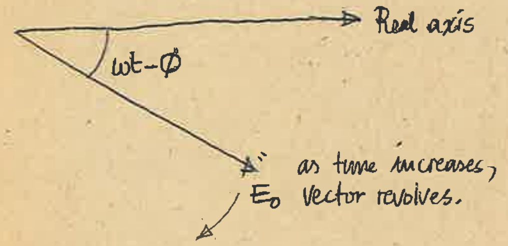
Time harmonic fields

rpf: real physical field
rpq: real physical quantity

let $E_{rpf} = E_0 \cos(\omega t - \phi) = Re\{E_0 e^{-j\phi} e^{j\omega t}\}$

we will call the quantity $E = E_0 e^{j\omega t - j\phi}$ as phasor

~~ωt~~$(-\phi)$: phase (E_0 is real)



$$\frac{\partial E_{rpf}}{\partial t} = -\omega E_0 \sin(\omega t - \phi)$$

$$\frac{\partial E_{rpf}}{\partial t} = Re\left\{\frac{\partial}{\partial t} E\right\} = Re\left\{j\omega E_0 e^{j\omega t - j\phi}\right\}$$

$$= Re\{j\omega E\}$$

in phasor domain : $\boxed{\begin{array}{l} \frac{\partial}{\partial t} \rightarrow j\omega \\ \frac{\partial^2}{\partial t^2} \rightarrow -\omega^2 \end{array}}$

So : $\frac{\partial E}{\partial t} = j\omega E$

$\frac{\partial^2 E}{\partial t^2} = -\omega^2 E$

– For the sake of simplicity let us agree that will drop writing $e^{j\omega t}$ all the time; but we know it should be considered.
We will need to include $e^{j\omega t}$ when it is required to pass the rpq.

if $e^{j\omega t}$ is assumed : $\frac{\partial}{\partial t} \rightarrow j\omega t$ EE Convention

if $e^{-j\omega t}$ is assumed : $\frac{\partial}{\partial t} \rightarrow -j\omega t$

Concept of complex permittivity :

$\nabla \times \vec{H} = \vec{J} + j\omega\epsilon\vec{E}$ where $\vec{D} = \epsilon\vec{E}$

Suppose that $\vec{J} = \sigma\vec{E}$ in a conducting medium

$\nabla \times \vec{H} = (\sigma + j\omega\epsilon)\vec{E} = j\omega\,\varepsilon_c\vec{E}$

$\varepsilon_c = \varepsilon + \frac{\sigma}{j\omega} = \varepsilon - j\frac{\sigma}{\omega}$: complex permittivity

So:

$\nabla \times \vec{H} = j\omega\,\varepsilon_c\vec{E}$ in a conducting medium

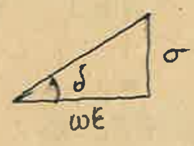
$\nabla \times \vec{H} = j\omega\,\epsilon\vec{E}$ in a non-conducting medium.

Definition : Relative complex permittivity (Complex ~~relative~~ dielectric constant)

$$\boxed{\varepsilon_{cr} = \frac{\varepsilon_c}{\varepsilon_o}} = \frac{\varepsilon}{\varepsilon_o} - j\frac{\sigma}{\omega\varepsilon_o}$$

$$\varepsilon_{cr} = \varepsilon_r - j\frac{\sigma}{2\pi f\frac{1}{36\pi}\times 10^{-9}} = \varepsilon_r - j\frac{18\sigma\times 10^9}{f}$$

Definition : loss tangent



$\tan\delta = \frac{\sigma}{\omega\varepsilon}$: loss tangent

$= \frac{\sigma}{\omega\varepsilon_o\varepsilon_r}$

if $\tan\delta$ is large : poor dielectric

if $\tan\delta$ is small : good dielectric

Complex vector operations :

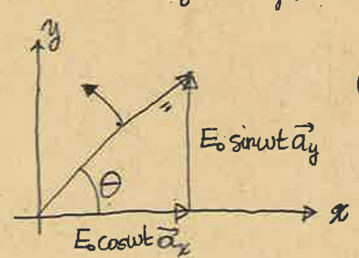
1 – $\vec{E} = \vec{E}_1 + \vec{E}_2 = (E_o\vec{a}_x + E_o e^{-j\frac{\pi}{2}}\vec{a}_y)\,e^{j\omega t}$

$\vec{E}_{rpf} = Re\{\vec{E}\} = E_o\cos\omega t\,\vec{a}_x + E_o\cos(\omega t - \frac{\pi}{2})\,\vec{a}_y$

$= E_o\cos\omega t\,\vec{a}_x + E_o\sin\omega t\,\vec{a}_y$

$|\vec{E}_{rpf}| = \sqrt{E_o^2\cos^2\omega t + E_o^2\sin^2\omega t}$

$= E_o \leftarrow$ fixed



$\tan\theta = \frac{\cancel{E_o}\sin\omega t}{\cancel{E_o}\cos\omega t} = \tan\omega t \Rightarrow \theta = \omega t \longrightarrow$ wave revolves.

18101980

E_{rpf} is not determined by space vector addition of phasor vector components of $\vec{E}$ in general.

If $\vec{E}_1$ and $\vec{E}_2$ are in phase ; Let ;

$\vec{E}_{rpf} = E_o e^{j\alpha}\vec{a}_x + q\vec{E}_o e^{j\alpha}\vec{a}_y$; (q and E_o are real)

we say that, $\vec{E}_1 = E_o e^{j\alpha}\vec{a}_x$

$\vec{E}_2 = qE_o e^{j\alpha}\vec{a}_y$ are in phase.

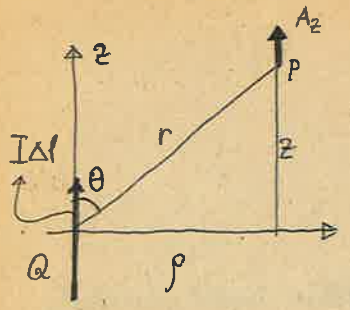
$\vec{E}_{rpf} = Re\{\vec{E}\} = (E_o\vec{a}_x + qE_o\vec{a}_y)\cos(\omega t + \alpha)$

$\vec{E} = (E_o\vec{a}_x + qE_o\vec{a}_y)\,e^{j\alpha}$

2 – If $\vec{E}\cdot\vec{a} = 0$ where $\vec{a}$ is a constant vector then $\vec{E}_{rpf}\cdot\vec{a} = 0$ holds.

3 – If $\vec{E}\times\vec{a} = 0$; then $\vec{E}_{rpf}\times\vec{a} = 0$ where $\vec{a}$ is a constant vector.

Example : Field of a time-harmonic current element (Hertz dipole)



$I\Delta l = I_0 e^{j\omega t}\Delta l$

to find $\vec{E}$ and $\vec{H}$, we use the method of auxiliary potentials.
$\vec{A}$ = vector potential.

$$\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$
$$\vec{B} = \nabla\times\vec{A}$$

$$\frac{\partial^2}{\partial t^2} = -w^2 \Rightarrow \nabla^2\vec{A} + \mu_0\varepsilon_0 w^2\vec{A} = -\mu_0\vec{J}$$ (at phasor quantity)

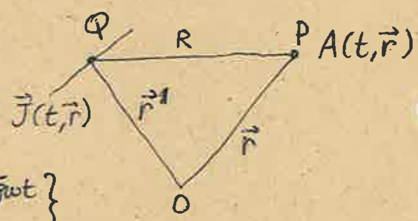
we define $\boxed{k_0 = \omega\sqrt{\varepsilon_0\mu_0}}$ propagation constant. (wave number)

$$k_0 = \frac{\omega}{\frac{1}{\sqrt{\varepsilon_0\mu_0}}} = \frac{\omega}{c} = \frac{2\pi f}{c}$$

we call the quantity $\lambda_0 = \frac{f}{c}$: wave length in free space.
$[\lambda_0]$: meters
$[k_0]$: 1/meters

$$k_0 = \frac{2\pi}{\lambda_0}$$

We had the solution for $\vec{A}(R,t)$ as : $\underset{\psi}{\vec{A}_{rpf}} = \int \frac{\mu_0\vec{J}_{rpf}(t-\frac{R}{c}\vec{r})}{4\pi R}$



$$\vec{A} = \int \frac{\mu_0\vec{J}_0(\vec{r}')e^{j\omega(t-R/c)}}{4\pi R}dr'$$ where $\vec{J}_{rpf}(t,r) = Re\{\underbrace{\vec{J}_0(\vec{r}')e^{j\omega t}}_{phasor}\}$

$\vec{r}' = 0$; $\vec{R} = \vec{r}$; $R = r$: $\int \vec{J}_0(\vec{r}')dr' = I\Delta l$

$$A_z(t,r) = \frac{\mu_0 I_0\Delta l e^{j\omega(t-R/c)}}{4\pi R} = \frac{\mu_0 I_0 e^{-jk_0 r}\Delta l\, e^{j\omega t}}{4\pi r}$$

(to be continued)

Electric and Magnetic field :

$\vec{B} = \nabla \times \vec{A}$

$B_\varphi = -\dfrac{\partial A_z}{\partial \rho} = -\dfrac{\partial A_z}{\partial r}\cdot\dfrac{\partial r}{\partial \rho}$

$\vec{A} = A_z(\rho, z)\vec{a}_z$ (in cylindrical coordinates)

$r = (\rho^2+z^2)^{1/2}$

$\dfrac{\partial r}{\partial \theta} = \dfrac{\rho}{r} = \dfrac{\cancel{r}\sin\theta}{\cancel{r}} = \sin\theta$

$B_\phi = -\sin\theta\,\dfrac{\partial A_z}{\partial r}$

$B_\phi = \dfrac{I\Delta l}{4\pi}\sin\theta\; jk_o\mu_o\,\dfrac{e^{-jk_or}}{r}\left(1+\dfrac{1}{jk_or}\right)$

let us consider all r such that $k_or \gg 1$ i.e $r \gg \lambda_o$

This range is called the far-field of the Hertz dipole.

$B_\phi \approx \dfrac{I\Delta l}{4\pi}\sin\theta\; jk_o\mu_o\,\dfrac{e^{-jk_or}}{r}\quad (r\gg\lambda_o)$

Calculating the electric field :

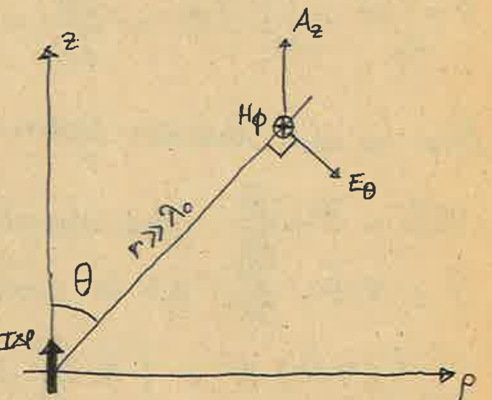
$\nabla\times\vec{B} = \mu_o(j\omega\varepsilon_o\vec{E})$

$j\omega\varepsilon_o\mu_o E_r = \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta B_\phi)$ (in spherical coordinates) $\hookrightarrow \propto \dfrac{e^{-jk_or}}{r^2}$

$j\omega\varepsilon_o\mu_o E_\theta = -\dfrac{1}{r}\dfrac{\partial}{\partial r}(rB_\phi) \;\propto\; jk_o\dfrac{e^{-jk_or}}{r} = jk_or\,\dfrac{e^{-jk_or}}{r^2}$

if $k_or \gg 1$ then $|E_r| \ll |E_\theta|$

$$\left.\begin{aligned} E_\theta &\approx \frac{I\Delta l}{4\pi}\sin\theta\, j\omega\mu_o\,\frac{e^{-jk_or}}{r}\\ H_\phi &\approx \frac{I\Delta l}{4\pi}\sin\theta\, jk_o\,\frac{e^{-jk_or}}{r}\end{aligned}\right\}\quad \begin{aligned}&k_or\gg1 \Rightarrow r\gg\lambda_o\\ &\text{since } k_o=\frac{2\pi}{\lambda_o}\end{aligned}$$



observations :

i) E_θ and H_ϕ are in phase in the far-field

ii) They are having the same space variations $\sin\theta\,\dfrac{e^{-jk_or}}{r}$

iii) $\dfrac{|E_\theta|}{|H_\phi|} = \dfrac{\omega\mu_o}{k_o} = \dfrac{\cancel{\omega}\mu_o}{\cancel{\omega}\sqrt{\varepsilon_o\mu_o}} = \sqrt{\dfrac{\mu_o}{\varepsilon_o}} = 120\pi \approx 377\,\Omega$ intrinsic impedance of free space Z_o

iv) E_θ, H_ϕ are tangent to a sphere with radius R.

the quantity $E_\theta\vec{a}_\theta = Z_o\,\vec{a}_r\times H_\phi\vec{a}_\phi$

$\vec{E} = Z_o\,\vec{a}_r\times\vec{H}$

$\vec{S} = \vec{E}\times\vec{H} = E_\theta H_\phi\,\underbrace{\vec{a}_\theta\times\vec{a}_\phi}_{\vec{a}_r}$

$\vec{S} = E_\theta H_\phi\,\vec{a}_r \longrightarrow$ Poynting's vector.

$\vec{a}_r$ is the direction of propagation.

23101980

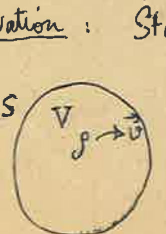
Energy in the electromagnetic field

Poynting's theorem : $\begin{cases}\text{time domain (instantaneous Poynting's th.)} & (1)\\ \text{Frequency domain (complex Poynting's th.)} & (2)\end{cases}$

(1) Time domain Poynting's theorem

(Declaration of the conservation of energy in the closed system)

Derivation : Start from Lorentz force



ρ : volume charge moving with velocity $\vec{v}$

$\vec{J}$: current density $(A/m^2) = \rho\vec{v}$

Let $\vec{E}$ and $\vec{B}$ be the electromagnetic field set up in V by $\vec{J}$

$d\vec{F} = \rho\,dV(\vec{E}+\vec{v}\times\vec{B})$ Lorentz force acting on ρdV = charge in $dV\,m^3$

This force $d\vec{F}$, delivers an instantenous power :

$$dP = \underset{\text{newton}}{d\vec{F}} \cdot \underset{m/sec}{\vec{v}} = \text{watts (power)} = \frac{\text{Newton} \times m}{sec} = \frac{\text{Joules}}{sec}$$

$$dP = \rho dV(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v} = \rho dV \vec{E} \cdot \vec{v}$$

Because $(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$

$\perp$

Since $\vec{J} = \rho\vec{v}$, $dP = \vec{J} \cdot \vec{E}\, dV$

If $\vec{E} \cdot \vec{J} > 0$ the field causes charge ρ to accelerate increasing its kinetic energy

If $\vec{E} \cdot \vec{J} < 0$ the field causes the charge to deccelabate and charge looses kinetic energy. This is also equivalent to the saying that, the charge delivers some of its energy to the field.

$$P = \int_V dP = \int_V \vec{E} \cdot \vec{J}\, dV = \frac{dW}{dt} = \text{power delivered to } \vec{J} \text{ in } V \text{ by the electromagnetic field.}$$

Now we use Maxwell's equations in V to analyze $\int_V \vec{E} \cdot \vec{J}\, dV = \frac{dW}{dt}$

$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ (Amp. Maxwell equation.)

$\vec{J} = \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t}$ will be substituted.

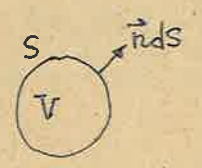
$$\int_V \vec{E}(\nabla \times \vec{H})\, dV - \int_V \vec{E} \frac{\partial \vec{D}}{\partial t} dV = \frac{dW}{dt}$$

Let us use $\nabla(\vec{E} \times \vec{H}) = \vec{H}(\nabla \times \vec{E}) - \underbrace{\vec{E}(\nabla \times \vec{H})}$

$$\int_V \vec{H}(\nabla \times \vec{E})\, dV - \int_V \nabla(\vec{E} \times \vec{H})\, dV - \int_V \vec{E} \frac{\partial \vec{D}}{\partial t} dV = \frac{dW}{dt}$$

Now use $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law)

Also apply divergence theorem to $\int_V \nabla(\vec{E} \times \vec{H})\, dV = \oint_S (\vec{E} \times \vec{H})\vec{n} \cdot dS$



$$\underbrace{\oint_V \vec{E} \cdot \vec{J}\, dV}_{\frac{dW_q}{dt}} + \underbrace{\oint_S (\vec{E} \times \vec{H})\vec{n}\, ds}_{\frac{dW_p}{dt}} + \oint_V \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} dV + \oint \vec{E} \frac{\partial \vec{D}}{\partial t} dV = 0$$

$$\frac{dW_q}{dt} + \frac{dW_p}{dt} + \frac{dW_s}{dt} = 0$$

$\frac{d}{dt}(W_q + W_p + W_s) = 0$ or $W_q + W_p + W_s =$ constant w.r.t time

for a linear medium: $\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t}(\frac{1}{2}\vec{E} \cdot \vec{D})$

Let us use $\vec{J} = \sigma(\vec{E} + \vec{E}_i)$ $\quad \vec{E}_i$: impressed field

$$\left\{ \underbrace{\int_V \vec{E}_i \cdot \vec{J}\, dv}_{\substack{\text{Power delivered}\\ \text{by the source}}} = \underbrace{\int_V \frac{J^2}{\sigma}\, dv}_{\substack{\text{Joule heat}\\ \text{loss}}} + \underbrace{\oint_S \vec{P}\cdot\vec{n}\, ds}_{\substack{\text{Radiation}\\ \text{loss}}} + \underbrace{\int_V \left(\vec{H}\,\frac{\partial \vec{B}}{\partial t} + \vec{E}\,\frac{\partial \vec{D}}{\partial t}\right) dv}_{\text{Stored power in } V} \right\}$$

For quasi-stationary case we neglect the displacement current;

$$\left|\frac{\partial \vec{D}}{\partial t}\right| \ll |\vec{J}|$$

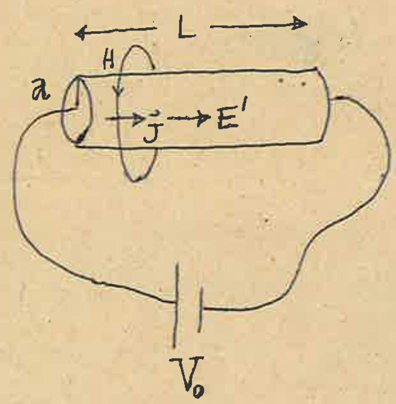
$$\nabla \times \vec{H} \approx \vec{J}$$

$$\int_V \vec{E}\cdot\vec{J}\, d\vec{v} + \oint_S (\vec{E}\times\vec{H})\, d\vec{s} + \int_V \vec{H}\,\frac{\partial \vec{B}}{\partial t}\, dv = 0$$

For the static case $\quad [\vec{E}\,(\text{induced}) = 0]$

$$\int_V \vec{E}'\cdot\vec{J}\, dv + \oint_S (\vec{E}'\times\vec{H})\, d\vec{s} = 0 \qquad E' : E_{\text{impressed}}$$

Example



$$E' = \frac{V_0}{L} \qquad J = \frac{I}{\pi a^2}$$

$$H_\phi = \frac{I}{2\pi r} \quad (r \geq a)$$

$$\vec{E}'\times\vec{H} = \frac{V_0}{L}\,\vec{a}_z \times \frac{I}{2\pi r}\,\vec{a}_\phi$$

$$= \frac{V_0 I}{2\pi a L}\,(-\vec{a}_r)$$

$$\int_S (\vec{E}'\times\vec{H})\, d\vec{s} = \frac{V_0 I}{2\pi a L}\cdot(-\vec{a}_r)\cdot(\vec{a}_r)\cdot 2\pi a L$$

$$= -V_0 I$$

$$\vec{E}'\cdot\vec{J} = \frac{V_0}{L}\,\vec{a}_z\,\frac{I}{\pi r^2}\cdot\vec{a}_z$$

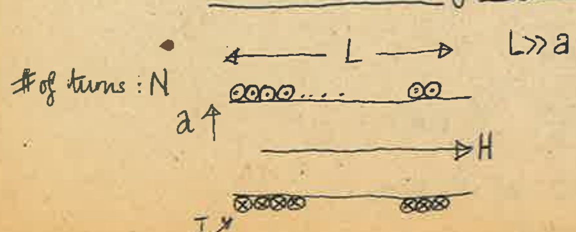
$$= \frac{V_0 I}{L\pi r^2}$$

$$\int_V \vec{E}\cdot\vec{J}\, dv = \int_0^a \frac{V_0 I}{L\pi r^2}\,\underbrace{2\pi r\, dr\, L}_{dV}$$

$$= V_0 I$$

28101980

Example for Poynting's thm. for Quasi stationary field

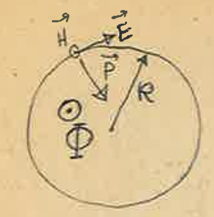


$$\boxed{H \approx \frac{NI}{L}}$$

$$B = \mu_0 \frac{NI}{L}$$

Let $I = I(t)$ (slowly varying with time)

Then E is induced



$$\Phi = \pi R^2 B$$

$$E = \frac{d\Phi/dt}{2\pi r} \quad \text{(Faraday's law)}$$

$$E = \frac{\mu_0 N}{L}\frac{r}{2}\frac{dI}{dt}$$

Poynting's vector

$$\vec{P} = \vec{E}\times\vec{H} \quad (W/m^2)$$

$$P = \frac{\mu_0 N}{L}\frac{r}{2}\frac{dI}{dt}\frac{NI}{L}(-\vec{a}_r)$$

$$\int_{S_r} \vec{P}\cdot d\vec{s} = P\cdot 2\pi r\cdot L = -\frac{\mu_0 N}{L}\frac{r}{2}\frac{dI}{dt}\frac{NI}{L}\cdot 2\pi r L$$

$\hookrightarrow$ due to $-\vec{a}_r\cdot d\vec{s}$

S_r Cylinder of length L radius r

$$\int_{V_r} \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\,dV = \frac{NI}{L}\mu_0\frac{N}{L}\frac{dI}{dt}\underbrace{\pi r^2 L}_{V_r}$$

V_r: Volume of S_r

we see that $\int_{S_r}\vec{P}\cdot d\vec{s} = \int_{V_r}\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\,dV$

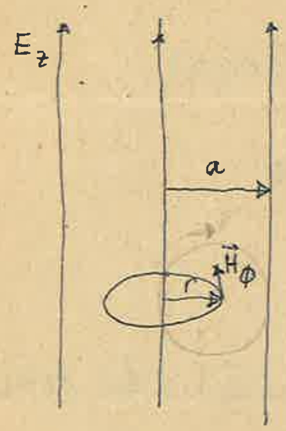
time rate of change of magnetic stored energy

which is the quasi stationary Poynting's theorem.

<u>Example</u> for complete time-varying case for Poynting's theorem

<u>Given</u>:

$$E_z = \frac{1}{2}pt^2 + q \qquad t: \text{time}, \quad p, q: \text{constants} \qquad (r \leq a)$$



Midterm question from early years

<u>Question</u>: a) Find all H and B

b) Demonstrate the Poynting's thm.

<u>Solution</u>:

a) Displacement current: $J_{Dz} = \varepsilon_0\frac{\partial E_z}{\partial t} = \varepsilon_0 pt \quad (r\leq a)$

$$2\pi r H_\phi = \pi r^2 J_{Dz} \quad \text{(Ampere's law)}$$

$$\left.\begin{aligned} H_\phi &= \varepsilon_0\frac{prt}{2} \quad (r\leq a) \\ 2\pi r H_\phi &= \pi a^2 J_{Dz} \quad (r\geq a) \\ H_\phi &= \varepsilon_0\frac{pa^2t}{2r} \quad (r\geq a)\end{aligned}\right\} H_\phi$$

Another way of finding H_ϕ could be using differential form of Ampere's law.

$$\nabla\times H_\phi\vec{a}_\phi = J_{Dz}\vec{a}_z = \varepsilon_0 pt\,\vec{a}_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = \varepsilon_0 pt$$

integrating we obtain

$$H_\phi = \varepsilon_0\frac{ptr}{2} + \frac{C_1}{r} \quad (r\leq a)$$

at $r=0$ $H_\phi = 0$ so $C_1 \equiv 0$

$$H_\phi = \varepsilon_0\frac{ptr}{2} \quad (r\leq a)$$

What is $\vec{E}$?

Since H_ϕ is time-varying, it produces a secondary $\vec{E}$ (E_z^1), which is different from the given field E_z.

$H_\phi(t) \xrightarrow[\text{Faraday's law.}]{} E_z^1$

We used this equation before, in order to find H_ϕ;

$$\nabla \times \vec{H}_\phi = \varepsilon_0 \frac{\partial E}{\partial t}$$

find ⟵ given

So we can't use again,

in order to find secondary Electric field, we use Faraday's law.

$$\nabla \times \vec{E}_z^1 = -\mu_0 \frac{\partial H_\phi}{\partial t}$$

find ⟵ given

So, we $\underset{\nabla \times E_z^1 \vec{a}_z}{\nabla \times \vec{E}_z^1} = -\frac{\partial E_z}{\partial r}\vec{a}_\phi = -\mu_0 \frac{\partial H_\phi}{\partial t}\vec{a}_\phi = -\mu_0 \varepsilon_0 \frac{pr}{2}\vec{a}_\phi$

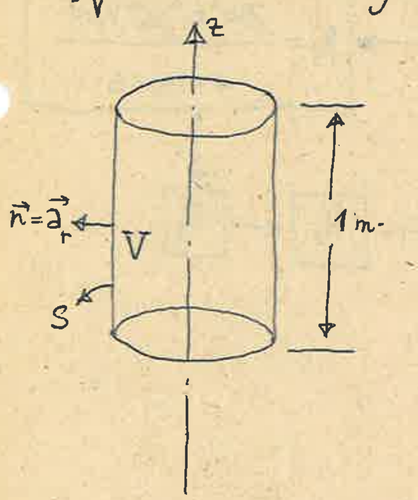
$$E_z^1 = \frac{\mu_0 \varepsilon_0 p r^2}{4} \quad r \leq a$$

there will be no secondary $\vec{H}$ because E_z^1 is not time varying.

$$\boxed{\begin{array}{l}\text{CONCLUSION:}\\ E_T = E_z + E_z^1\\ H_T = H_\phi\end{array}}$$

instantaneous Poynting's theorem:

$$\int_V \vec{J}\cdot\vec{E}\cdot dv + \frac{d}{dt}\int (w_e + w_m)\,dV + \oint_S (\vec{E}_T \times \vec{H})\,ds = 0$$



$$w_e = \frac{1}{2}\varepsilon_0 (E_z + E_z^1)^2$$

$$w_m = \frac{1}{2}\mu_0 \left(\frac{\varepsilon_0 p t}{2}\right)^2 r^2$$

$$dV = 2\pi r\,dr\cdot 1 = 2\pi r\,dr$$

$$\vec{E} \times \vec{H} = (\vec{E}_z + E_z^1) \times \vec{H}$$

$$= (\vec{E}_z + \vec{E}_z^1) H_\phi (-\vec{a}_r)$$

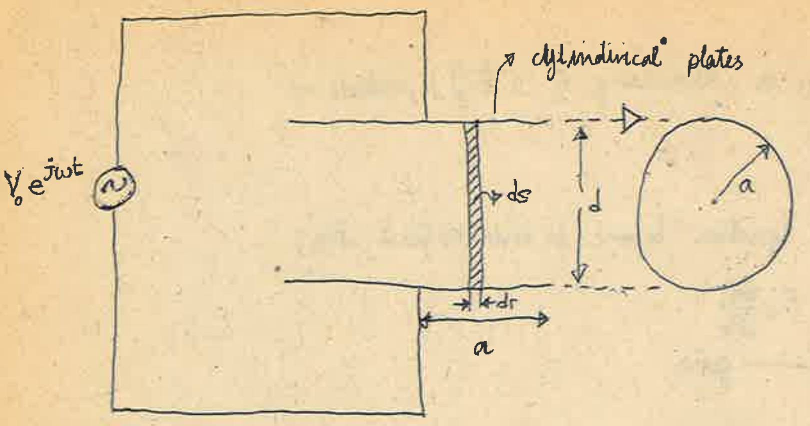
$$d\vec{s} = \vec{n}\,ds = 2\pi a \cdot 1 \cdot \vec{a}_r$$

$\vec{J} = 0$ (no conduction current)

19706

04119980

Another example for time-varying fields, in a cylindrical region; (time-harmonic excitation)



<u>Question</u>: Find E and H in the cylinder using Maxwell's eq.

<u>Hint!</u>: Use an iterative method

$$\boxed{E_0 = \frac{V_0}{d}} \rightarrow \boxed{\begin{aligned} \nabla \times \vec{H} &= j\omega\varepsilon_0 \vec{E} \\ 2\pi r H_1 &= j\omega\varepsilon_0 \frac{V_0}{d}\pi r^2 \end{aligned}}$$

$$\boxed{H_1 = \frac{j\omega\varepsilon_0 V_0 r}{2d}}$$

$$\boxed{\oint \vec{E}\cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} d\vec{s}}$$

$$\boxed{\begin{aligned} E(r+dr)d - E(r)d &= j\omega\mu_0 H(r)\underbrace{dr\,d}_{ds} \\ \frac{dE}{dr} = \frac{E(r+dr)-E(r)}{dr} &= j\omega\mu_0 \frac{j\omega\varepsilon_0 V_0}{2d} r \end{aligned}} \rightarrow \boxed{E_1 = \frac{-\omega^2\varepsilon_0\mu_0 V_0}{4d} r^2}$$

$$\boxed{\begin{aligned} &2\pi r H_2 = j\omega\varepsilon_0 \frac{-\omega^2\varepsilon_0\mu_0}{4d}\int_0^r r^2 2\pi r dr = I_D \\ &\text{total displacement current in } \pi r^2 \text{ area} \\ &I_D = j\omega\varepsilon_0 \int_0^r E_1 2\pi r dr \end{aligned}}$$

$$\boxed{H_2 = -\frac{j\omega^3\varepsilon_0^2\mu_0 V_0}{16d} r^3}$$

$$\boxed{\frac{dE_2}{dr} = j\omega\mu_0 \frac{-j\omega^3\varepsilon_0^2\mu_0 V_0}{16d} r^3} \rightarrow \boxed{E_2 = \frac{\omega^4\varepsilon_0^2\mu_0^2 V_0 r^4}{4\times 16d}} \rightarrow \boxed{H_3 = \frac{j\omega^5\varepsilon_0^3\mu_0^2 V_0 r^5}{4\times 16d\times 6}}$$

$$\infty \leftarrow \boxed{H_3} \leftarrow \boxed{E_3}$$

$$E_{TOTAL} = \frac{V_0}{d} - \frac{V_0}{d}\frac{\omega^2\varepsilon_0\mu_0}{4} r^2 + \frac{V_0}{d}\frac{\omega^4\varepsilon_0^2\mu_0^2 r^4}{4\times 16} - \frac{V_0}{d}\frac{\omega^6\varepsilon_0^3\mu_0^3}{4\times 6\times 6\times 16} r^6 \dots$$

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \qquad k_0 = \frac{\omega}{c}$$

$$E_T = \frac{V_0}{d}\left[1 - \left(\frac{k_0 r}{2}\right)^2 + \frac{\left(\frac{k_0 r}{2}\right)^4}{(2!)^2} - \frac{\left(\frac{k_0 r}{2}\right)^6}{(3!)^2} + \cdots\right]$$

A special function called <u>Bessel's function</u> = $J_n(x)$ where $n = 0, 1, 2 \dots$ x = real; called argument satisfies the following differential equation:

$$x^2 \frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0$$

n is called the order of $J_n(x)$

$$R = J_n(x) \qquad J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{x}{2}\right)^{2m+n}}{m!(m+n)!}$$

In this example we can write:

$$E_T = \frac{V_0}{d} J_0(k_0 r)$$

Alternative method: Consists of obtaining a single equation for E.

$$\nabla\times\vec{E} = -j\omega\mu_0\vec{H} \qquad \vec{H} = H_\phi \vec{a}_\phi$$
$$\nabla\times\vec{H} = j\omega\epsilon_0\vec{E} \qquad \vec{E} = E_z\vec{a}_z$$

$$\nabla\times\nabla\times\vec{E} = -j\omega\mu_0\nabla\times H = -j\omega\mu_0(j\omega\epsilon_0\vec{E})$$
$$\underbrace{\nabla\nabla\vec{E}}_{0} - \nabla^2\vec{E} = \omega^2\epsilon_0\mu_0\vec{E}$$
$$\nabla^2 E_z + k_0^2 E_z = 0 \qquad (k_0^2 = \omega^2\epsilon_0\mu_0)$$

$$E_z = E_z(r)$$
$$\nabla^2 E_z = \frac{1}{r}\frac{d}{dr}\left(r\frac{dE_z}{dr}\right) = \frac{1}{r}\left(\frac{dE_z}{dr} + r\frac{d^2E_z}{dr^2}\right)$$
$$\nabla^2 E_z + k_0^2 E_z = \frac{d^2E_z}{dr^2} + \frac{1}{r}\frac{dE_z}{dr} + k_0^2 E_z = 0$$

let us define: $k_0 r \equiv x$ and multiply by x^2.

$$x^2\frac{d^2E_z}{dx^2} + x\frac{dE_z}{dx} + x^2 E_z = 0 \longrightarrow$$ Bessel's differential eq. for $n=0$

$$E_z = \underbrace{K J_0(x) = K J_0(k_0 r)}_{} = \frac{V_0}{d} J_0(k_0 r)$$

What is the value of K? when $k_0 = 0$ (static case) $\boxed{E_z = \frac{V_0}{d} \text{ and } J_0(0) = 1}$

finding the $\vec{H}$:

$$\nabla\times\vec{E}_T = -j\omega\mu_0\vec{H}_T$$
$$\boxed{\nabla\times E_{Tz}\vec{a}_z} = -j\omega\mu_0 H_{T\phi}\vec{a}_\phi$$
$$= \frac{\partial E_{Tz}}{\partial r}$$

$$H_{T\phi} = -\frac{1}{j\omega\mu_0}\frac{dE_{Tz}}{dr} = -\frac{k_0}{j\omega\mu_0}\left(\frac{V_0}{d}\right)\underbrace{\frac{dJ_0(x)}{dx}}_{-J_1(x)}$$

$$\boxed{H_{T\phi} = +\frac{k_0}{j\omega\mu_0}\left(\frac{V_0}{d}\right)J_1(k_0 r)}$$

You could obtain the series expansion
$$H_{T\phi} = H_1 + H_2 + H_3 + \ldots\ldots$$

COMPLEX POYNTING'S THEOREM (for time harmonic fields in phasor notation)

(1) $\nabla\times\vec{H} = \vec{J}' + \vec{J} + j\omega\epsilon\vec{E}$ where J' impressed (external) current density

$\vec{J} = \sigma\vec{E}$ induced current density

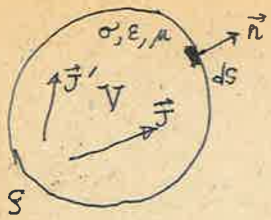
$\vec{J}' = \sigma\vec{E}'$

(2) $\nabla\times\vec{E} = -j\omega\mu\vec{H}$

Now consider: $\nabla(\vec{E}\times\vec{H}^*) \equiv \vec{H}^*\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H}^*)$

$$\equiv -j\omega\mu\vec{H}\cdot\vec{H}^* - (\sigma - j\omega\epsilon)\vec{E}\cdot\vec{E}^*$$

Integrate over a closed surface S



$$\frac{1}{2}\oint_S (\vec{E}\times\vec{H}^*)\,d\vec{s} = -j\omega\int_V \left(\frac{1}{2}\mu\vec{H}\cdot\vec{H}^* - \frac{1}{2}\varepsilon\vec{E}\vec{E}^*\right)dV - \int_V \frac{1}{2}\sigma\vec{E}\vec{E}^*\,dV - \frac{1}{2}\int_V \vec{E}\,\vec{J}'^*\,dV$$

we have also $\vec{J} = \sigma\vec{E} + \sigma\vec{E}'$, $\vec{E} = \frac{\vec{J}}{\sigma} - \vec{E}'$

Let us define $W_{e,av} \equiv \frac{1}{2}\left(\frac{1}{2}\varepsilon\vec{E}\cdot\vec{E}^*\right) = \frac{1}{4}\varepsilon\vec{E}\vec{E}^*$

$W_{m,av} \equiv \frac{1}{2}\left(\frac{1}{2}\mu\vec{H}\vec{H}^*\right) = \frac{1}{4}\mu\vec{H}\vec{H}^*$

time-averaged stored electric and magnetic energy densities. (Joules/m^3)

$$\underbrace{\int_V \frac{1}{2}\vec{E}'\cdot\vec{J}^*\,dv}_{\text{time averaged power delivered by the impressed source (field } \vec{E}')} = \oint_S \frac{1}{2}(\vec{E}\times\vec{H}^*)\,d\vec{s} + \int_V \frac{1}{2}\frac{\vec{J}\cdot\vec{J}^*}{\sigma}\,dV + \underbrace{2j\omega\int_V (W_{m,av} - W_{e,av})\,dV}_{\text{note that purely imaginary}}.$$

We call $\vec{P}^* = \frac{1}{2}\vec{E}\times\vec{H}^*$ as complex poynting's vector

$\oint_S \vec{P}^*\,d\vec{s}$ shows time averaged power crossing the surface S.

$\frac{1}{2}\int \frac{\vec{J}\cdot\vec{J}^*}{\sigma}\,dv$ shows the time averaged power delivered to the conductivity σ of V (Joule heat loss in V)

$j\omega\int (W_{m,av} - W_{e,av})\,dv$ shows time averaged stored (energy/second) power in V

If we want to express the time av. power (over a period) using phasor quantities:

$P = \frac{1}{2}VI^*$

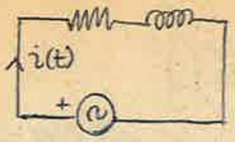
$P_{av} = Re\left\{\frac{1}{2}VI^*\right\} = \frac{1}{2}V_0 I_0$

We can split complex Poynting's vector into two parts:

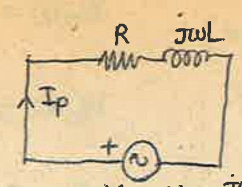
1. $\int_V Re\left\{\frac{1}{2}\vec{E}'\cdot\vec{J}^*\right\}dV = \oint_S Re\{\vec{P}^*\}\,d\vec{s} + \int_V \frac{1}{2}\frac{|J|^2}{\sigma}\,dV$ ⟶ REAL POWER FLOW EQ.

($\vec{J}$ and $\vec{J}^*$ are in phase)

2. $\int_V Im\left\{\frac{1}{2}\vec{E}'\cdot\vec{J}^*\right\}dV = -\oint_S Im\left\{\vec{P}^*\right\}d\vec{s} + \omega\int_V (W_{av,m} - W_{e,av})\,dv$ ⟶ REACTIVE POWER FLOW EQ.



in phasor $\longrightarrow$



$V = V\cos(\omega t + \phi)$ $\quad V_p = V_0 e^{j\omega t + j\phi}$

$V_L^p = j\omega L J_p$ (leads I_p)

$$I_p = \frac{V_0 e^{j\phi + j\omega t}}{R + j\omega L} = I_0 e^{j\alpha + j\omega t}$$

$$I_0 = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}, \quad \alpha = \phi - \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$P(t) = V(t) I(t) = V_0 I_0 \cos(\omega t + \phi)\cos(\omega t + \alpha)$$

period of $P(t)$ is $\frac{T}{2}$ where $T = \frac{2\pi}{\omega}$

$$P_{av}(t) = \frac{1}{T/2}\int_0^{T/2} P(t)\,dt = \frac{1}{2} V_0 I_0 \cos(\phi - \alpha)$$ we can write this also as $P_{av}(t) = \frac{1}{2}\mathrm{Re}\{V_p I_p^*\}$

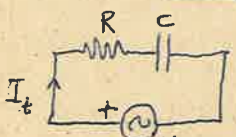
$$\frac{1}{2} V_p I_p^* = \frac{1}{2} R I_p I_p^* + j\omega L \frac{1}{2} I_p I_p^*$$

$\mathrm{Re}\left\{\frac{1}{2} V_p I_p^*\right\} = \frac{1}{2} R I_p I_p^* = \frac{1}{2} R I_0^2 \longrightarrow$ Joule heat loss (time averaged)

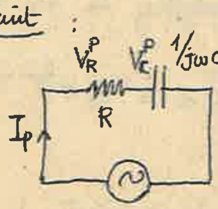
$\mathrm{Im}\left\{\frac{1}{2} V_p I_p^*\right\} = \omega L \frac{1}{2} I_0^2 = 2\omega\left(\frac{1}{4} L I_0^2\right)$ $\quad W_{m,av} = \frac{1}{4} L I_0^2$ time averaged stored energy

Consider now RC circuit:

06111980



$\longrightarrow$



$V_p = V_0 e^{j\omega t + j\phi}$ $\quad I_p = I_0 e^{j\omega t + j\alpha}$

$$\alpha = \tan^{-1}\frac{1}{\omega RC} + \phi \qquad I_0 = \frac{V_0}{\left(R^2 + \frac{1}{\omega^2 C^2}\right)^{1/2}}$$

$P(t) = \mathrm{Re}\{V_p I_p\}$

Real

$(P(t))_{AV} = \frac{1}{2}\mathrm{Re}\{V_p I_p^*\}$ Real power flow

$$\frac{1}{2} V_p I_p^* = \frac{1}{2} R I_p I_p^* - j\frac{1}{2\omega C} I_p I_p^* = \frac{1}{2} R I_p I_p^* - j2\omega \quad W_{e,av}$$

$W_{e,av} = \frac{1}{4} C V_c^p V_c^{p*}$ (Av stored energy in C)

$P(t)_{av} = \frac{1}{2} R I_p I_p^*$ (Joule heat loss)

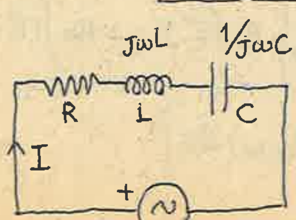
$$\underbrace{\mathrm{Im}\left\{\tfrac{1}{2} V_p I_p^*\right\}} = -2\omega W_{e,av} = \frac{1}{2} V_0 I_0 \sin(\phi - \alpha) = -\frac{1}{2} V_0 I_0 \sin\left(\tan^{-1}\frac{1}{\omega RC}\right)$$

Reactive power flow from the generator to the circuit which goes to the C.

If flow = 0 then no change of stored energy with time occurs.

Flow $\rightarrow$ $-\underbrace{2\omega\,(\text{Av. stored energy})}_{\text{av. time rate of change of energy.}}$

Take up now an RLC circuit:



$V_0\cos(\omega t + \phi) \longrightarrow V_0 e^{j\omega t + j\phi}$

$$I_p = I_0 e^{j\alpha + j\omega t}$$

$$I_0 = \frac{V_0}{\left[R^2 + (\omega L)^2 + \left(\frac{1}{\omega C}\right)^2\right]^{1/2}}$$

$$\alpha = \phi - \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}$$

$\mathrm{Re}\left\{\frac{1}{2} V_p I_p^*\right\} = \frac{1}{2} R I_p I_p^*$

$\mathrm{Im}\left\{\frac{1}{2} V_p I_p^*\right\} = 2\omega\left(W_{m,av} - W_{e,av}\right)$

total instantaneous stored energy:

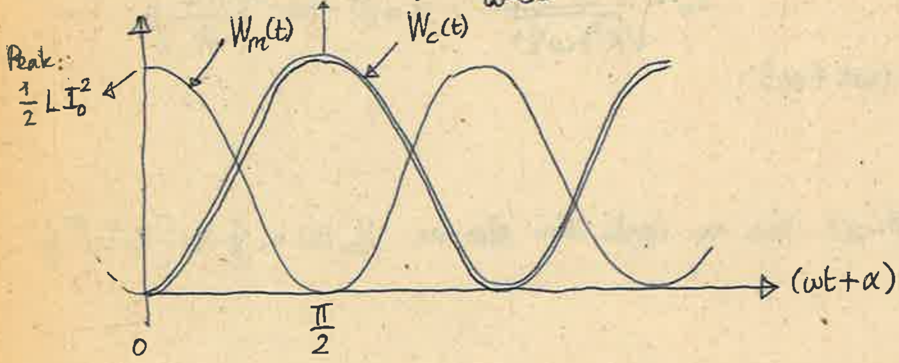
$$W_T(t) = W_m(t) + W_e(t)$$

$$W_T(t) = \underbrace{\frac{1}{2} L I_0^2 \cos^2(\omega t+\alpha)}_{W_m(t)} + \underbrace{\frac{1}{2} C \frac{I_0^2}{\omega^2 C^2} \sin^2(\omega t+\alpha)}_{W_e(t)}$$

$$W_m(t) = \frac{1}{2} L \left[Re\{I_p\} \right]^2$$

$$W_e(t) = \frac{1}{2} C \left[Re\{V_c^p\} \right]^2$$

$$V_c^p = I_p \frac{1}{j\omega C} = \frac{I_0}{\omega C} e^{j\omega t + j\alpha - j\frac{\pi}{2}}$$

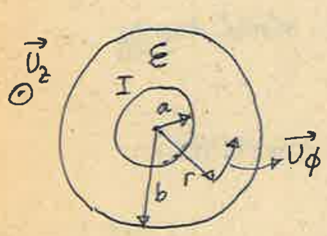


If we have $\omega L = \frac{1}{\omega C}$

$W_T(t) = \frac{1}{2} L I_0^2$, $\alpha = \phi$., $V_L^p + V_c^p = 0$

$Im\{\frac{1}{2} V_p I_p^*\} = 0$ RESONANCE CASE

Example : Power flow in a coaxial cable



Cross section of cable

$$H_\phi = \frac{I_0 e^{j\omega t - jkz}}{2\pi r}$$

$$E_r = \frac{V_0 e^{j\omega t - jkz}}{r \ln \frac{b}{a}}$$

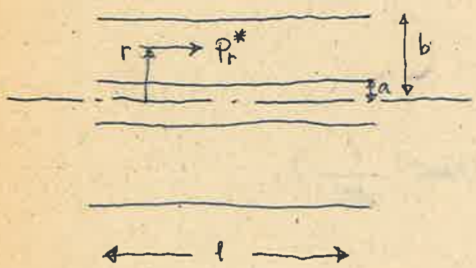
These are waves (because of the presence of the factors $e^{j\omega t - jkz}$) propagating in $+z$ direction.

$$\int_a^b E_r dr = V_0 e^{j\omega t - jkz}$$

Let us calculate $P^* = \frac{1}{2} \vec{E} \times \vec{H}^*$ (Complex Poynting's Vector)

$$\vec{P}^* = \frac{1}{2} E_r H_\phi^* \vec{u}_z$$

$$\vec{P}_r^* = \frac{1}{2} \frac{V_0}{r \ln \frac{b}{a}} \cdot \frac{I_0}{2\pi r} = \frac{V_0 I_0}{4\pi \ln \frac{b}{a}} \cdot \frac{1}{r^2}$$

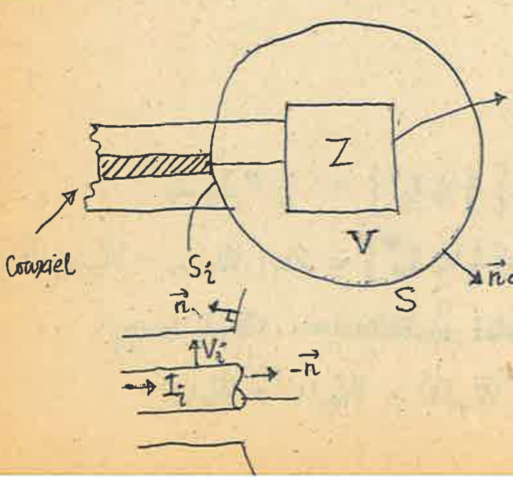


If we apply complex Poynting's theorem for the volume V in the cylinder of length l between inner and outer conductors. we obtain ;

$$\int_{S1} P_{r,1}^* ds = \int_{S} P_{r,2}^* ds$$

$$\int_{S1} P_{r,1}^* ds = \frac{V_0 I_0}{4\pi \ln \frac{b}{a}} \int_a^b \frac{1}{r^2} \underbrace{2\pi r dr}_{ds} = \frac{1}{2} V_0 I_0 \leftarrow \text{complex average Power.}$$

Example 1:



→ a passive electromagnetic system (antenna, absorber, circuit, another transmission line etc)

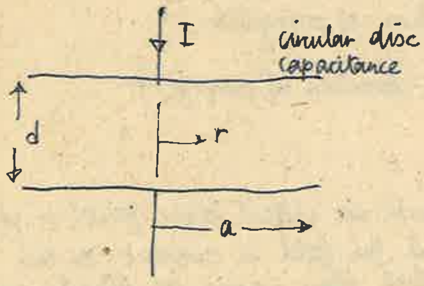
Let $V = Z I_i$ Z = impedance of the system $= R + jX$

Show that : $R = \frac{1}{|I|^2} \left\{ Re \int_V \vec{E} \cdot \vec{J} dv + Re \int_{S-S_i} (\vec{E} \times \vec{H}^*) ds - 4\omega \, Im \int_V (W_{m,av} - W_{e,av}) dV \right\}$

Continued on the next page ⇨

$$X = \frac{1}{|I|^2}\left\{ \mathrm{Im}\int_V \vec{E}\,\vec{J}^*\,dv + \mathrm{Im}\int_{S-S_i} (\vec{E}\times\vec{H}^*)\vec{n}\,ds + 4\omega\,\mathrm{Re}\int_V (W_{m,av} - W_{e,av})\,dv \right.$$

Apply the formulas to :



and show that $C \cong \dfrac{\pi a^2 \varepsilon_0}{d}$

$L \cong \dfrac{\mu_0 d}{8\pi}$

Example 2 :

Using the formulas of the radiation of a Hertz dipole calculate the total radiated real power using complex Poynting's vector

Hint : Take a sphere of any radius $r \gg \lambda$

Idealized Waves
(Plane waves)

In phasor form :

$\vec{E} = \vec{E}_0\, e^{j\omega t - jkz + j\phi}$ here $\vec{E}_0$ is a real constant vector.

$= E_0\, e^{j\omega t - jkz + j\phi}\,\vec{a}_x$

$k_0 = \omega\sqrt{\varepsilon_0\mu_0}$: wave number in free space or propagation constant

$\vec{H} = H_0\, e^{j\omega t - jkz + j\phi}\,\vec{a}_y$ where E_0 and H_0 are real constant quantities.

$\nabla\times\vec{E} = -j\omega\mu_0\vec{H}$ ← Maxwell's eqn. (Faraday)

$\nabla\times\vec{E} = e^{-jk_0z}\vec{a}_x = E_0\nabla\times e^{-jk_0z}\vec{a}_x = E_0\left(e^{-jk_0z}\,\nabla\times\vec{a}_x^{\,0} + \nabla e^{-jk_0z}\times\vec{a}_x\right)$ $\boxed{\nabla\times(c\vec{C}) = c\nabla\times\vec{C} + \nabla c\times\vec{C}}$

$= E_0\nabla e^{-jk_0z}\times\vec{a}_x = E_0\left(-jk_0\, e^{-jk_0z}\vec{a}_z\right)\times\vec{a}_x = -jk_0E_0\, e^{-jk_0z}\vec{a}_y$

cross

$\vec{a}_z\times\vec{a}_x = \vec{a}_y$ $\quad -jk_0E_0\, e^{-jk_0z + j\omega t + j\phi}\,\vec{a}_y = -j\omega\mu_0\vec{H}_0$

$\vec{H} = \dfrac{k_0}{\omega\mu_0}E_0\, e^{j\omega t - jk_0z + j\phi}\,\vec{a}_y$

Thus $H_0 = \dfrac{k_0}{\omega\mu_0}E_0 = \dfrac{E_0}{Z_0}$ $\quad Z_0 = \left(\dfrac{k_0}{\omega\mu_0}\right)^{-1} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 120\pi\ \Omega$.

Now start from $H = H_0\, e^{j\omega t - jkz + j\phi}\,\vec{a}_y$ and find $\vec{E}$ using Ampere's law. ⇦ Exercise

$\nabla\times\vec{H} = j\omega\varepsilon\vec{E}$

Let us check that $\nabla\vec{E} = 0$ and $\nabla\vec{H} = 0$:

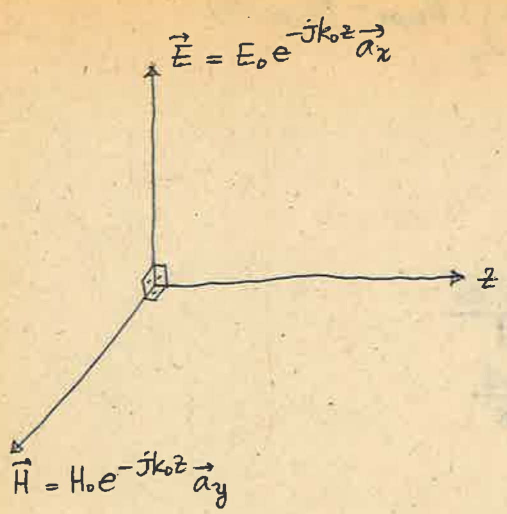
$\nabla\vec{E} = E_0\nabla\left(e^{-jkz}\vec{a}_x\right) = E_0\, e^{-jk_0z}\,\nabla\vec{a}_x^{\,0} + E_0\left(-jk_0 e^{-jk_0z}\right)\vec{a}_z\cdot\vec{a}_x^{\,0} = \underline{\underline{0}}$

$\left\{\nabla\cdot(c\vec{C}) = c\nabla\vec{C} + \nabla c\,\vec{C}\right\}$

$c = e^{-jk_0z}$
$\vec{C} = \vec{a}_x$

Also verify that : $\nabla^2\vec{E} + k_0^2\vec{E} = 0$

$\nabla^2\vec{H} + k_0^2\vec{H} = 0$



Properties of plane waves

1 _ Constant phase surfaces are plane, perpendicular to the direction of propagation.
2 _ $\vec{E}$ and $\vec{H}$ are in the equiphase surface that is $\vec{E} \perp \vec{H}$ and (E and H) $\perp$ direction of propagation.
3 _ $\vec{E} \times \vec{H}$ points into the direction of propagation
4 _ $\frac{|\vec{E}|}{|\vec{H}|} = Z$
5 _ Equi-phase surfaces which are called wave-front (or phase front) are infinitely large and the field is constant on this surface. That is an ideal plane wave carries an infinite amount of energy (power)

ie $|\vec{P}^*| = \frac{1}{2} E_0 H_0 = \frac{1}{2} Z_0 H_0^2$

Show that:

$\vec{E} = E_0 e^{-j\vec{k_0}\cdot\vec{r}}\vec{a}$
$\vec{a}$: constant unit vector
$\vec{r}$: position vector
$\vec{k_0}$: $k_0 \cdot \vec{u}$
$\vec{u}$: constant unit vector

} is plane wave

$\vec{H} = \frac{\vec{u} \times \vec{E}}{Z}$ $\vec{u} = \frac{\vec{k}}{|\vec{k}|}$

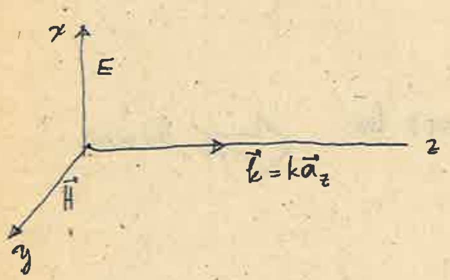
equiplane surface is $\vec{k_0}\cdot\vec{r}$ = constant and the direction of propagation is $\vec{u}$

11111980

$\vec{r} = x\vec{a_x} + y\vec{a_y} + z\vec{a_z}$
$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$
$\vec{k} = k_x\vec{a_x} + k_y\vec{a_y} + k_z\vec{a_z}$
we have obviously $\sqrt{k_x^2 + k_y^2 + k_z^2} = |\vec{k}|$

Example:

Set $\vec{k} = k\vec{a_z}$, $k_z = k$, $k_x = k_y = 0$
$\vec{a} = \vec{a_x}$



Remember always $\vec{E} \times \vec{H}$ is always directed in $\vec{u}$ direction

$\vec{E} = E_0\vec{a_x} e^{-jkz + j\omega t}$
$\vec{H} = H_0\vec{a_y} e^{-jkz + j\omega t}$
$H_0 = \frac{E_0}{Z}$
$\vec{u} = \vec{a_z}$

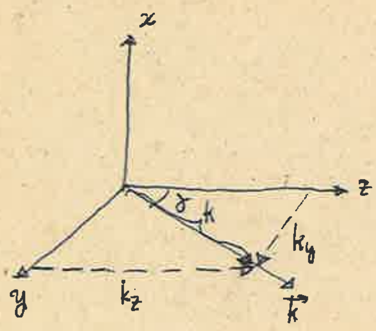
$E_{rpf} = \vec{a_x} E_0 \cos(\omega t - kz)$
$H_{rpf} = \vec{a_y} \frac{E_0}{Z} \cos(\omega t - kz)$

Example: $k_z = -k$

$\vec{E} = \vec{a_x} E_0 e^{jkz + j\omega t}$ what is $\vec{u}$?
$\vec{k} = -k\vec{a_z}$
$\vec{u} = -\vec{a_z}$
$\vec{H} = -\vec{a_y} H_0 e^{jkz + j\omega t}$

Definition : Phase velocity : v_p

Example :



Let $\vec{k}$ lie in yz-plane making an angle γ with the z axis. γ is given. Write down the complete expressions for the field of a plane wave, propagating in $\vec{k}$ direction.

$$k = k_y\vec{a}_y + k_z\vec{a}_z$$

$$k_y = k\sin\gamma$$

$$k_z = k\cos\gamma$$

$$e^{-j\vec{k}\cdot\vec{r}} = e^{-jk_y y - jk_z z} = e^{-jk\sin\gamma y - jk\cos\gamma z}$$

Let us choose (It may be given to you) $\vec{E} = E_o\,\vec{a}_x\,e^{-j\vec{k}\cdot\vec{r}}$

related $\vec{H}$: $\vec{H} = \frac{\vec{k}}{|k|}\times\frac{\vec{E}}{Z} = \frac{1}{kZ}\,\vec{k}\times\vec{E}$

$$\vec{H} = \frac{1}{kZ}(k_y\vec{a}_y + k_z\vec{a}_z)\times\vec{a}_x E_o e^{-j\vec{k}\cdot\vec{r}}$$

$$\vec{H} = \frac{1}{kZ}(-k_y\vec{a}_z + k_z\vec{a}_y)E_o e^{-j\vec{k}\cdot\vec{r}}$$

Let us calculate phase velocity of this wave :

v_p : we write $e^{-jk_y y - jk_z z + j\omega t}$

$$\cancel{\boxed{e^{-j\omega(t - \frac{\vec{k}\cdot\vec{r}}{\omega})}}}$$

$$e^{+j\omega(t - \frac{\vec{k}\cdot\vec{r}}{\omega})}$$

we define v_z : phase velocity in z-direction

$$\boxed{v_z = \frac{\omega}{k_z}} \qquad v_y = \frac{\omega}{k\sin\gamma} = \frac{v}{\sin\gamma}$$

$$\boxed{v_y = \frac{\omega}{k_y}} \qquad v_z = \frac{\omega}{k\cos\gamma} = \frac{v}{\cos\gamma}$$

where $\boxed{v_p = \frac{\omega}{k}}$

$\Downarrow$ or v

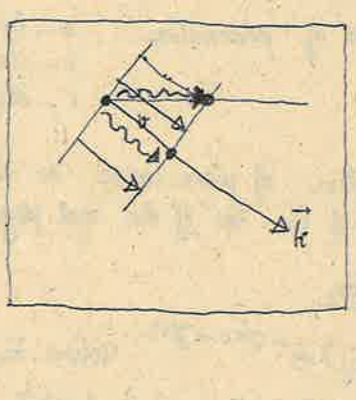
we define v_p as the velocity by which an observer should travel in the direction of propagation $\vec{u}$ to see a constant phase.

we can define :

$$\boxed{\lambda_z = \frac{2\pi}{k_z}} > \lambda = \frac{2\pi}{k}$$

$$\boxed{\lambda_y = \frac{2\pi}{k_y}} > \lambda$$

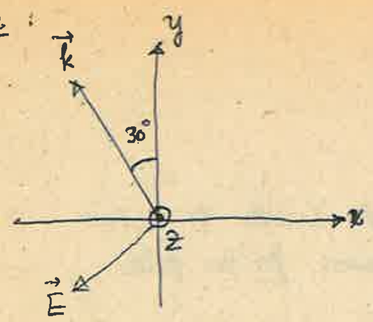
$\lambda_x =$



Complex Poynting's Vector :

$$\vec{P}^* = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}E_oH_o(\vec{a}\times\vec{b}) = \frac{1}{2}E_oH_o\vec{u} \quad \text{watts/m}^2$$

Example:



$\vec{H}$ is along $\vec{z}$ direction, ω is given

a) Find out the complete expression for $e^{-j\vec{k}\cdot\vec{r}}$, $\vec{H}$ and $\vec{E}$

b) Find phase velocity v, v_y, v_x

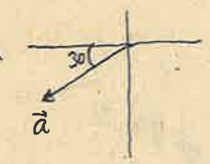
$\cos 30° = \frac{\sqrt{3}}{2}$

$\sin 30° = \frac{1}{2}$

$$-j\vec{k}\cdot\vec{r} = -jky\frac{\sqrt{3}}{2} - jkx\frac{1}{2} = -j\frac{k}{2}(x+\sqrt{3}y)$$

$$\vec{H} = H_0\vec{a}_z e^{-j\frac{k}{2}(x+\sqrt{3}y)+j\omega t}$$

$$\vec{E} = Z \cdot H_0 \vec{a} \cdot e^{-j\frac{k}{2}(x+\sqrt{3}y)+j\omega t}$$



$$\vec{a} = -\cos 30° \vec{a}_x - \sin 30 \vec{a}_y$$

$$\vec{a} = -\frac{1}{2}(\sqrt{3}\vec{a}_x + \vec{a}_y)$$

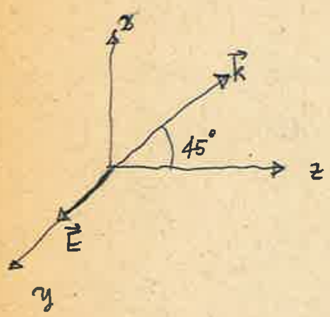
$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu_0}} \qquad v_y = \frac{v}{\cos 30°} = \frac{2v}{\sqrt{3}} \qquad v_x = \frac{-v}{\sin 30°} = -2v$$

Example:

$$\vec{E} = E_0\vec{a}_{xy} e^{-jkCz - jkDy + j\omega t}$$

$$k = \omega\sqrt{\varepsilon\mu_0}$$

Find C and so that wave propagates in xz-plane making an angle of 45° with z-axis



$C^2 + D^2 = 1$

$C = \cos 45° = \frac{\sqrt{2}}{2} D$

$k_z = kC$ $\quad k_z^2 + k_y^2 = k^2 \rightarrow C^2 + D^2 = 1$

$k_y = kD$

Find $\vec{H}$:

Find v_x, v_z

Polarization of Plane Waves

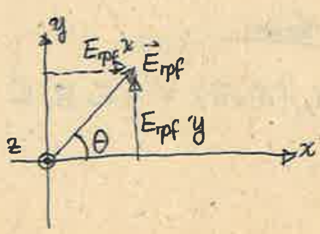
There are 3 types of polarization:
a – linear
b – circular
c – elliptical

Definition – Polarization of plane wave is the curve in the space which is the locus of the points of the tip of the real physical electrical field vector.

① Linear Polarization

$\vec{E} = (E_1\vec{a}_x + E_2\vec{a}_y)e^{-jkz + j\omega t}$ where E_1 and E_2 are real.

$$\vec{E}_{rpf} = \underbrace{E_1\cos(\omega t - kz)\vec{a}_x}_{E_{rpf}\ x\text{ component}} + \underbrace{E_2\cos(\omega t - kz)\vec{a}_y}_{E_{rpf}\ y\text{ component}}$$



$$\tan\theta = \frac{E_{rpf}\,y}{E_{rpf}\,x} = \frac{E_1}{E_2}$$

So θ is fixed in space

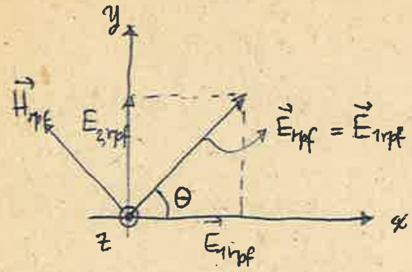
13.11.1980

19

Continue with the polarization of a plane wave:

Definition of polarization: the direction of E_{rpf}

Consider: $\vec{E} = (E_1\vec{a}_x + E_2\vec{a}_y)e^{-jkz+j\omega t}$



$\vec{E}_{rpf} = \vec{E}_{1rpf} + \vec{E}_{2rpf} = (E_1\vec{a}_x + E_2\vec{a}_y)\cos(\omega t - kz)$

$\tan\theta = \frac{E_2}{E_1}$: Constant (linear polarization)

$|E_{rpf}| = \sqrt{E_1^2 + E_2^2}$

Circular polarization

Now consider: $\vec{E} = (E_0\vec{a}_x + jE_0\vec{a}_y)e^{-jkz+j\omega t}$

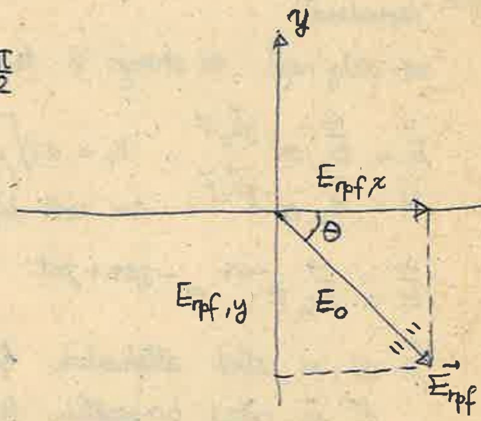
$\vec{a}_x \cdot \vec{E}_0 = E_0 e^{-jkz+j\omega t}$

$\vec{a}_y E = jE_0 e^{-jkz+j\omega t} = E_0 e^{-jkz+j\omega t + j\frac{\pi}{2}}$

$j = e^{j\frac{\pi}{2}}$.

$$\vec{E}_{rpf} = E_0\cos(\omega t - kz)\vec{a}_x + E_0\cos(\omega t - kz + \tfrac{\pi}{2})\vec{a}_y$$

$$= \underbrace{E_0\cos(\omega t - kz)\vec{a}_x}_{E_{rpf,x}} - \underbrace{E_0\sin(\omega t - kz)\vec{a}_y}_{E_{rpf,y}}$$



left hand circular polarization rotates in CW direction

$$|\vec{E}_{rpf}| = \sqrt{E_{rpf\,x}^2 + E_{rpf\,y}^2}$$
$$= \sqrt{E_0^2\cos^2(\ ) + E_0^2\sin^2(\ )}$$
$$= \sqrt{E_0^2(\underbrace{\cos^2(\ ) + \sin^2(\ )}_{1})}$$
$$= E_0$$

$$\tan\theta = \frac{E_{rpf,y}}{E_{rpf,x}} = \frac{\cancel{E_0}\sin(\omega t - kz)}{\cancel{E_0}\cos(\omega t - kz)} = \tan(\omega t - kz)$$

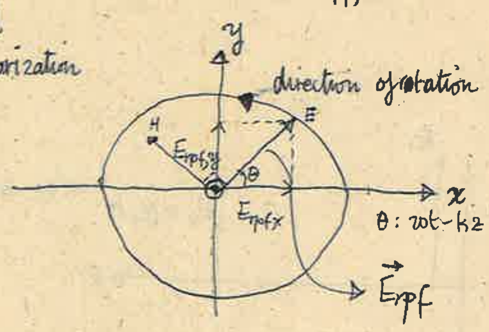
$\theta = \omega t - kz$ ← time dependent.

$\frac{d\theta}{dt} = \omega$ ← velocity

Let now $\vec{E} = (E_0\vec{a}_x - jE_0\vec{a}_y)e^{-jkz+j\omega t}$

$$\vec{E}_{rpf} = \underbrace{E_0\cos(\omega t - kz)\vec{a}_x}_{E_{rpf,x}} + \underbrace{E_0\sin(\omega t - kz)\vec{a}_y}_{E_{rpf,y}}$$

Right hand circular polarization

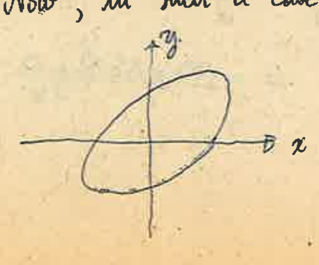


direction of rotation (~~CCW~~) Clockwise ([illegible])

More general elliptical Polarization:

$\vec{E} = (\vec{E}_1\vec{a}_x + E_2 e^{j\phi}\vec{a}_y)e^{-jkz+j\omega t}$ ϕ : $\frac{\pi}{2}$ or any other phase.

Now, in such a case we have tilted ellipse



<u>Example</u> : $\vec{E}_1 = E_0(\vec{a}_x + j\vec{a}_y)\, e^{-jkz}$ L.H.C

$\vec{E}_2 = E_0(\vec{a}_x - j\vec{a}_y)\, e^{-jkz}$ R.H.C

$\vec{E}_1 + \vec{E}_2 = 2E_0 \vec{a}_x e^{-jkz}$ ← linear polarization

<u>Example</u> : $\vec{E}_1 = E_0(-\vec{a}_x - j\vec{a}_y)\, e^{-jkz}$ L.H.C

$(-\vec{a}_x + j\vec{a}_y)\, e^{-jkz}$ R.H.C

$(j\vec{a}_x + \vec{a}_y)\, e^{-jkz}$ R.H.C

$(-j\vec{a}_x + \vec{a}_y)\, e^{-jkz}$ L.H.C

<u>Plane Waves in Conducting Media ($\varepsilon_r, \sigma, \mu_r$)</u>

With $e^{j\omega t}$ dependence.

we only need to change ε to $\varepsilon_c = \varepsilon + \frac{\sigma}{j\omega}$

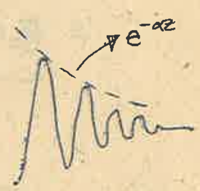
$\vec{E} = \vec{E}_a e^{-j\vec{k}_c \cdot \vec{r}}$ $k_c = \omega\sqrt{\mu\varepsilon_c}$: Complex

$\vec{H} = \vec{H}_a e^{-j\vec{k}_c \cdot \vec{r}}$ we will define $\gamma = jk_c = j\omega\sqrt{\mu\varepsilon_c} = j\omega\sqrt{\mu\varepsilon - j\frac{\mu\sigma}{\omega}} = \alpha + j\beta$

$\vec{E} = \vec{E}_a e^{-\alpha z} e^{-j\beta z + j\omega t}$

So α is called attenuation factor

β is called propagation factor. $E_{rpf} = E_a e^{-\alpha z} \cos(\omega t - \beta z)$



Solution of $\alpha + j\beta = j\omega\sqrt{\mu\varepsilon - j\frac{\mu\sigma}{\omega}}$

The solution is $\beta = \pm\, \omega \left\{ \frac{\mu\varepsilon}{2}\left[1 + \left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{1/2}\right]\right\}^{1/2}$

$$\alpha = \omega \left\{ \frac{\mu\varepsilon}{2}\left[-1 + \left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{1/2}\right]\right\}^{1/2}$$

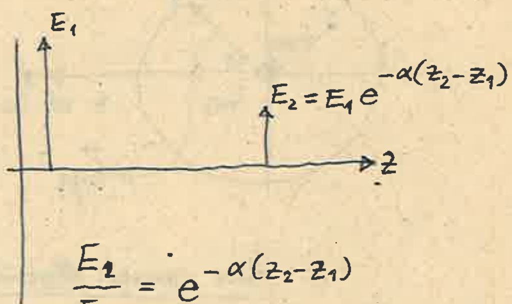
$\vec{E} = Z_c \vec{H} \times \vec{a}_z$

$Z_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon - j\frac{\sigma}{\omega}}}$

phase velocity $v = \frac{\omega}{\beta}$

$\lambda = \frac{v}{\omega} 2\pi = \frac{2\pi}{\beta}$

α has the dimension called <u>Nepers/m</u> or <u>decibels</u> :



$\frac{E_2}{E_1} = e^{-\alpha(z_2 - z_1)}$

$20 \log \frac{E_2}{E_1} = 20 \log e^{-\alpha\overbrace{(z_2 - z_1)}^{\Delta z}}$

$= -20\,\alpha\,\Delta z \log_{10} e \quad \left(\frac{E_2}{E_1}\right)_{db}$

CASE I Good dielectric

$\omega\varepsilon \gg \sigma$

Displacement current >> conduction current

for good dielectric case we use these approximations:

$$\boxed{\alpha \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}}$$

$$\boxed{\beta \cong \omega\sqrt{\mu\varepsilon}\left(1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right) > k = \omega\sqrt{\mu\varepsilon}}$$

$$\frac{1}{1+x} = 1 - x + \cancel{x^2} = 1-x \qquad |x| \ll 1$$

$$v = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon}}\left(1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right) < v_o = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\lambda = \frac{2\pi}{\beta} = \lambda_o\left(1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right) < \lambda_o$$

$$\boxed{Z_c = \underbrace{\sqrt{\frac{\mu}{\varepsilon}}}_{Z_o}\left(1+j\frac{\sigma}{2\omega}\right)}$$

CASE II Good conductor

$\omega\varepsilon \ll \sigma$

$|\vec{J}_D| \ll |\vec{J}|$

18111980

$$\gamma = jk_c = j\omega\sqrt{\mu\varepsilon - j\frac{\omega\sigma}{\omega}} \cong j\omega\sqrt{-j\frac{\mu\sigma}{\omega}} = j\omega\sqrt{\frac{\mu\sigma}{\omega}}\sqrt{e^{-j\frac{\pi}{2}}} = j\omega\sqrt{\frac{\mu\sigma}{\omega}}\, e^{-j\frac{\pi}{4}}$$

$\gamma = \alpha + j\beta$ and $\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$

we define $\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}}$ (meters) → SKIN DEPTH

$$\boxed{\alpha = \beta = \frac{1}{\delta}}$$

$\beta \gg k_o = \omega\sqrt{\varepsilon_o\mu_o}$

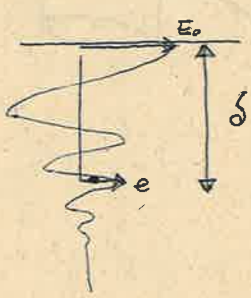
$\lambda \ll \lambda_o = \frac{2\pi}{k_o}$

$v = \frac{\omega}{\beta} = \omega\delta = \sqrt{\frac{2\omega}{\mu\sigma}} \ll c$

δ is small → $\frac{1}{\delta} = \beta$ → large

$\lambda = \frac{2\pi}{\beta}$ → small.

$$\boxed{\frac{E_0}{e} \rightarrow \delta}$$



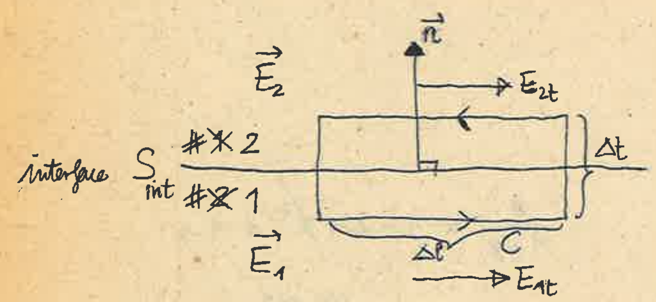
$$Z_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon - j\frac{\sigma}{\omega}}}$$

$$\approx \sqrt{\frac{\mu}{-j\frac{\sigma}{\omega}}}$$

$$\boxed{Z_c \cong \frac{1+j}{\sigma\delta}}$$

Boundary condition for time varying case:

① Continuity of tangential $\vec{E}$ field. (It is always continuous)



We apply Faraday's law:

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_{mag.\ flux}}{dt}$$

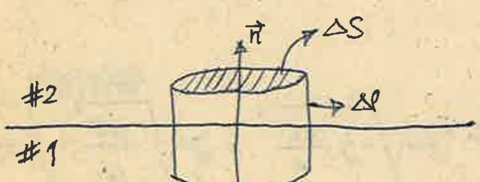
to C: $E_{1t}\Delta l + E_{1n}\Delta t - E_{2t}\Delta l - E_{2n}\Delta t = -\frac{\partial}{\partial t}(\vec{B}\cdot\Delta l\cdot\Delta t)$

let $\Delta t \to 0$

$E_{1t} - E_{2t} = 0$

$$\boxed{E_{1t} = E_{2t}}$$

② Continuity of normal Magnetic field (It is always continuous)



Apply $\nabla\vec{B} = 0$

$\oint \vec{B}\cdot d\vec{S} = 0$

to a pill-box surface

$B_{2n}\cdot\Delta S = B_{1n}\Delta S$

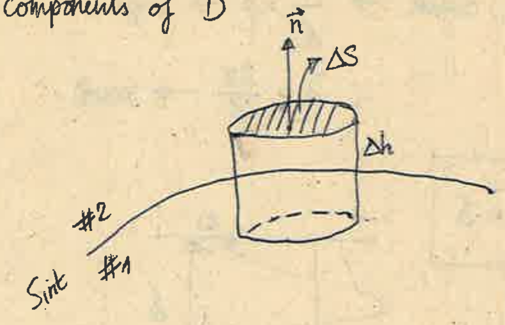
So $\boxed{B_{2n} = B_{1n}}$

"Recall that $\nabla\vec{B} = 0$ can be obtained from $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$" So "$B_n$ is continuous" is equivalent to "E_t is continuous" and Vice Versa.

③ B.C for the normal components of $\vec{D}$

$\nabla\vec{D} = \rho_v$

$\oint \vec{D}\cdot d\vec{S} = Q_{TOT\ in\ S}$



So when $\Delta h \to 0$ $\quad Q_{TOT\ in\ S} \to \rho_s\cdot\Delta S$

ρ_s is the surface charge density.

$$\boxed{D_{2n} - D_{1n} = \rho_s}$$

when $\rho_s \neq 0$?:

$\rho_s \neq 0$ if
a) One of the media is perfectly conducting
Then ρ_s is either induced or source
b) On S_{int} a ρ_s as a source is placed.
c) $\sigma_1 \neq \sigma_2$ and $\frac{\varepsilon_1}{\sigma_1} \neq \frac{\varepsilon_2}{\sigma_2}$ i.e $\tau_1 \neq \tau_2$

General case

$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = \nabla(\sigma\vec{E}) + \frac{\partial\rho}{\partial t}$

multiply by ε / $\boxed{\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = \sigma\nabla\vec{E} + \vec{E}\nabla\sigma + \frac{\partial\rho}{\partial t}}$ $\longrightarrow$ $\varepsilon\frac{\partial\rho}{\partial t} + \sigma\rho = E(\sigma\nabla\varepsilon - \varepsilon\nabla\sigma)$

$\nabla\cdot\vec{D} = \rho$

$\nabla\cdot(\varepsilon\vec{E}) = \rho$ $\qquad$ $\varepsilon\frac{\partial\rho}{\partial t} + \sigma\rho = \sigma\vec{J}\cdot\nabla\frac{\varepsilon}{\sigma} \longrightarrow \rho = \rho_0 e^{-\frac{t}{\tau}} + \vec{J}\nabla\frac{\varepsilon}{\sigma}$

σ / $\boxed{\varepsilon\nabla\vec{E} + E\nabla\varepsilon = \rho}$

If $\nabla \frac{\varepsilon}{\sigma} \neq 0$

Then $\rho \to 0 + \vec{J}\cdot \nabla \frac{\varepsilon}{\sigma}$

But if $\nabla \frac{\varepsilon}{\sigma} = 0$ Then $\rho \to 0$

we conclude that if $\tau_1 = \tau_2$ $\quad -\frac{\partial \rho_s}{\partial t} = \frac{\rho_s}{\tau_1}$

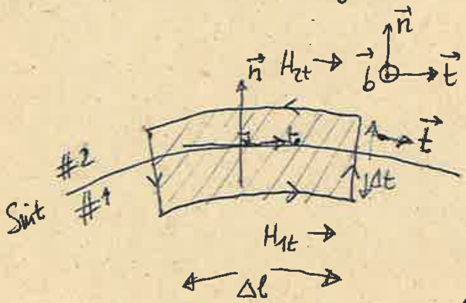
$$\boxed{\tau_2 J_{2n} - \tau_1 J_{1n} = \rho_s}$$

So if $\rho_s = 0$ it means (#1, 2) are dielectrics $\sigma_1 ; \sigma_2 = 0$ no ρ_s is placed as a source.
or $\tau_1 = \tau_2$

In that case:

$$\left\{ \begin{array}{l} D_{2n} = D_{1n} \\ \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n} \end{array} \right\}$$

④ B.C about the tangential magnetic fields:



We use Ampere's law:

$$\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S} + \int_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{S}$$

$$H_{1t}\Delta l - H_{2t}\Delta l = J_b \cdot \Delta l \cdot \Delta t + \frac{\partial D_b}{\partial t}\Delta l \cdot \Delta t$$

let $\Delta t \to 0$

$\frac{\partial D_b}{\partial t} =$ finite so $\frac{\partial D_b}{\partial t}\Delta l \Delta t \to 0$

$$H_{1t} - H_{2t} = J_b \Delta t$$

$$\boxed{H_{1t} - H_{2t} = J_{bs}}$$

Surface current density

$\boxed{BC\#4 \Leftrightarrow BC\#3}$

$J_{sb} \neq 0$ if

① #1 or #2 is perfectly conducting
Then J_{sb} is either induced or source.

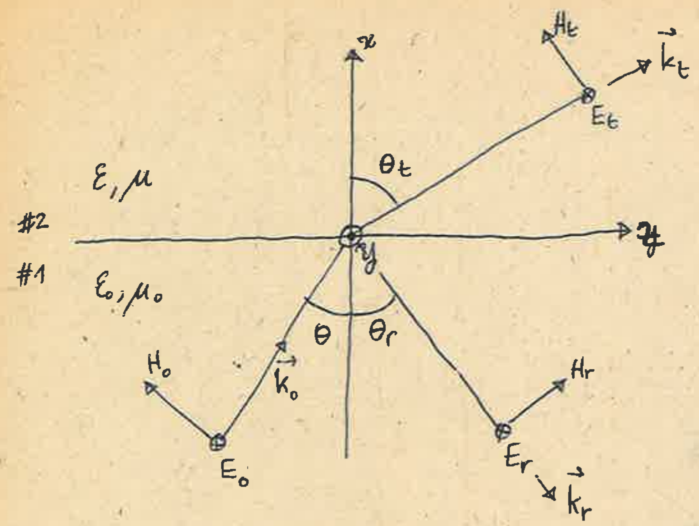
② J_{sb} is placed as a source.

③ $\tau_1 \neq \tau_2$

 25111980 (24)

Reflection and Refraction of Plane Waves

(i) Horizontal Polarization : (linear) (Media are lossless) TE Waves.



incident wave ($\vec{E}_0, \vec{H}_0, \vec{k}_0, \theta$) given

$x-z$ plane is called the plane of incidence (P.O.I)

$E \perp$ PoI

$E \parallel yz$-plane (interface)

$$\vec{E}_i = E_0(-\vec{a}_y)\, e^{-jk_0 \sin\theta z - jk_0 \cos\theta x}$$ incident Wave

$$\vec{k}_0 = k_x\vec{a}_x + k_y\vec{a}_y = k_0\cos\theta\, \vec{a}_x + k_0 \sin\theta\, \vec{a}_z$$

$$\vec{r} = x\vec{a}_x + z\vec{a}_z$$

$$\vec{k}_0 \cdot \vec{r} = (k_0\cos\theta)x + (k_0 \sin\theta) z.$$

$$\vec{E}_r = E_r(-\vec{a}_y)\, e^{-j\vec{k}_r\cdot\vec{r}} = E_r(-\vec{a}_y)\, e^{-jk_0 \sin\theta_r z + jk_0\cos\theta_r x}$$

$$\vec{k}_r = -k_0 \cos\theta_r \vec{a}_x + k_0 \sin\theta_r \vec{a}_z$$

$$\vec{E}_t = E_t(-\vec{a}_y)\, e^{-j\vec{k}_t\cdot\vec{r}} = E_t(-\vec{a}_y)\, e^{-jk_t\cos\theta_t x - jk_t \sin\theta_t z}$$

$$\vec{k}_t = k_t \cos\theta_t \vec{a}_x + k_t \sin\theta_t \vec{a}_z \qquad k_t = |\vec{k}_t| = \omega\sqrt{\varepsilon\mu}$$

$$\vec{H}_i = \frac{\vec{k}\times\vec{E}_i}{k Z_0}$$

we use the boundary conditions on interface ($x=0$)

(1) B.C : $E_{tangential}$ (TOTAL) is continuous

when $x=0$ $\vec{E}_0 + \vec{E}_r = \vec{E}_t$

$$E_0 e^{-jk_0 \sin\theta z} + E_r e^{-jk_0 \sin\theta_r z} = E_t e^{-jk_t \sin\theta_t z}$$

Since this equation should be satisfied for all values of z the exponents should be equal i.e. we should have

$$k_0 \sin\theta = k_0 \sin\theta_r = k_t \sin\theta_t$$

Phases of three waves at $x=0$ should match. (Phase matching)

$$k_{0z} = k_{rz} = k_{tz}$$

we conclude that :

kine-matical results

1. $\theta_r = \theta$
2. $k_0 \sin\theta = k_t \sin\theta_t \longrightarrow \sin\theta_t = \frac{k_0}{k_t}\sin\theta = \frac{\omega\sqrt{\varepsilon_0\mu_0}}{\omega\sqrt{\varepsilon\mu}}$ $\quad \sin\theta_t = \frac{\sin\theta}{\sqrt{\varepsilon_r\mu_r}} = \frac{\sin\theta}{n}$

$n = \sqrt{\varepsilon_r\mu_r}$: index of refraction ≥ 1

Snell's law.

results about amplitudes : (Dynamical results)

(1) BC : $E_0 + E_r = E_t$

(2) BC : H_t continuous ($J_s = 0$) or D_n continuous ($\rho_s = 0$) $D_n = 0$

So we use $H_{tangent}$ (TOT) continuous

$$H_o\cos\theta - H_r\cos\theta = H_t\cos\theta_t$$

$$\boxed{\frac{E_o}{Z_o}\cos\theta - \frac{E_r}{Z_o}\cos\theta = \frac{E_t}{Z_t}\cos\theta_t}$$

$$\frac{E_t}{E_o} = \Gamma_T = \frac{2Z_t\cos\theta}{Z_t\cos\theta + Z_o\cos\theta_t} = \text{Transmission coefficient}$$

$$\frac{E_r}{E_o} = \Gamma_R = \frac{Z_t\cos\theta - Z_o\cos\theta_t}{Z_t\cos\theta + Z_o\cos\theta_t} = \text{Reflection coefficient.}$$

Divide by $\cos\theta\cos\theta_t$

$$\Gamma_T = \frac{2\dfrac{Z_t}{\cos\theta_t}}{\dfrac{Z_t}{\cos\theta_t} + \dfrac{Z_o}{\cos\theta}}$$

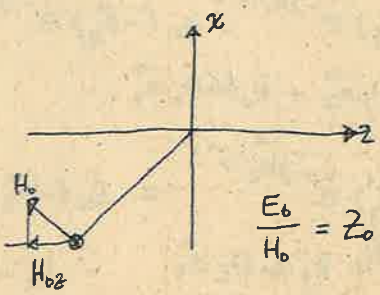
let us define $Z_{ox} = \dfrac{Z_o}{\cos\theta}$

$$Z_{tx} = \frac{Z_t}{\cos\theta_t}$$

$$\Gamma_T = \frac{2Z_{tx}}{Z_{tx} + Z_{ox}} \qquad \Gamma_R = \frac{Z_{tx} - Z_{ox}}{Z_{tx} + Z_{ox}}$$

For the dynamical results, Oblique incidence is equivalent to a normal incidence with $Z_o \to Z_{ox}$ and $Z_t \to Z_{tx}$

What are Z_{ox}, Z_{tx} ?



$$\frac{E_o}{H_o} = Z_o \qquad \frac{E_o}{H_{oz}} = Z_{ox}$$

Special cases:

(i) Set $\theta = 0$ Normal incidence

$\theta_r = 0$

$\theta_t = 0$

$$\Gamma_T = \frac{2Z_t}{Z_t + Z_o} \qquad \Gamma_R = \frac{Z_t - Z_o}{Z_t + Z_o}$$

(ii) When $\sqrt{\varepsilon_r\mu_r} = n \to \infty$ $\quad \varepsilon_r \to \infty$

In that case #2 is equivalent to a perfectly conducting medium.

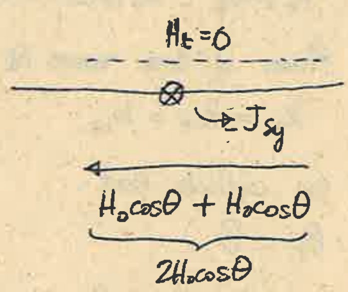
$$\Gamma_T = \frac{0}{Z_{ox}} = 0$$

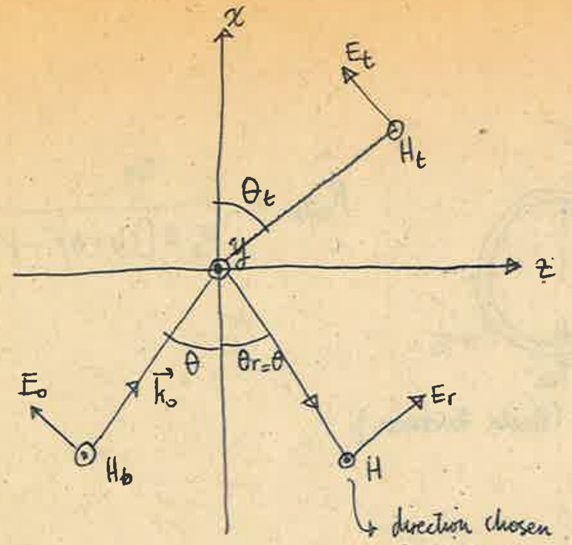
$$Z_{tx} = \frac{Z_t}{\cos\theta} = \sqrt{\frac{\mu_t}{\varepsilon_r}}\,\frac{1}{\cos\theta} \longrightarrow 0$$

$$\Gamma_R = -1$$

$$H_o\cos\theta + H_o\cos\theta = J_s$$

$$\boxed{J_{sy} = -2H_o\cos\theta}$$





direction chosen is arbitrary (Exact direction is found from equations)

Similarly: $\theta_r = \theta$

$k_o \sin\theta = k_t \sin\theta_t$

$$\sin\theta_t = \frac{\sin\theta}{n_{(n)}}$$

$$E_{tz} = E_t \cos\theta_t$$

$$E_o/H_o = Z_o$$

$$E_{oz} = E_o \sin\theta$$

$$\frac{-E_{oz}}{H_o} = Z_{ox} = Z_o \cos\theta$$

Z_{ox} : directional wave impedance

$$E_{tz} = E_t \cos\theta_t$$

$$\frac{-E_{tz}}{H_t} = Z_{tx} = Z_t \cos\theta$$

$$\frac{E_t}{E_o} = \Gamma_T^{TM} = \frac{2Z_t \cos\theta}{Z_{ox} + Z_{tx}}$$

$$\frac{E_r}{E_o} = \frac{H_r}{H_o} = \Gamma_R^{TM} = \frac{Z_{ox} - Z_{tx}}{Z_{ox} + Z_{tx}}$$

$$\Gamma^{TM}_{T(H\,field)} = \frac{H_t}{H_o} = \frac{E_t/Z_t}{E_o/Z_o} = \frac{E_t}{E_o}\frac{Z_o}{Z_t}$$

$$= \frac{2Z_{ox}}{Z_{ox} + Z_{tx}}$$

Total reflection.

Transmission lines

04121980

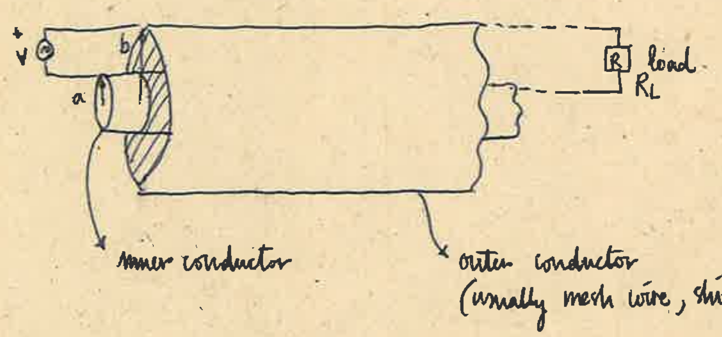
i- Coaxial cable
ii- Two wire line
iii- Wave guide
iv- Optical waveguides
v- Surface wave structures
vi- In general, any physical system (natural or man-made) capable of supporting traveling waves

Example of transmission lines.

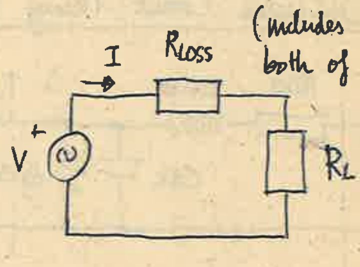
We will deal with physical transmission lines for which we can define a voltage V and a current I

1- Coaxial Cable
2- Two wire line

are two examples.

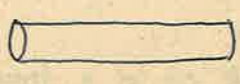


I and -I are the currents in conductors



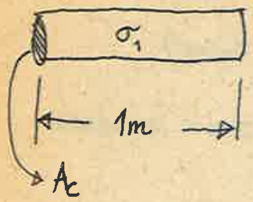
(includes total resistance both of inner and outer conductors)

equivalent CCt.



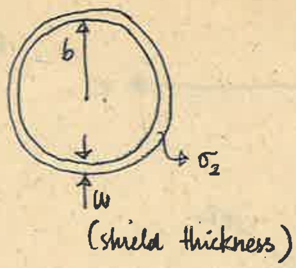
Series Resistances

inner conductor :



$$R_{\text{inner cond.}} = \frac{1m}{\sigma_1 A_c}$$

outer conductor



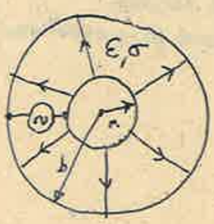
$$R_{\text{outer}} = \frac{1m}{\sigma_2 \pi \left[(b+w)^2 - b^2\right]}$$

Parallel (Shunt) Resistance :

$$J_{loss} = \sigma E = \sigma \frac{V}{(\ln \frac{b}{a}) r}$$

$$I = \int_0^{2\pi} J_{loss} \underset{m}{1} r \, d\theta = 2\pi r J_{loss} = \frac{2\pi\sigma}{\ln \frac{b}{a}} V$$

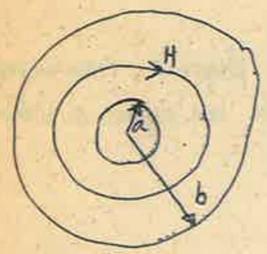
$$\frac{V}{I_{loss}} = \frac{\ln \frac{b}{a}}{2\pi\sigma} = R_{shunt}. \quad (1m.)$$



Capacitance (Farads/m)

$$C = \frac{q_l}{V} = \frac{2\pi\epsilon}{\ln \frac{b}{a}} \quad \text{Farads/m}$$

Inductance (Henry/m)



$$H = \frac{I}{2\pi r}$$

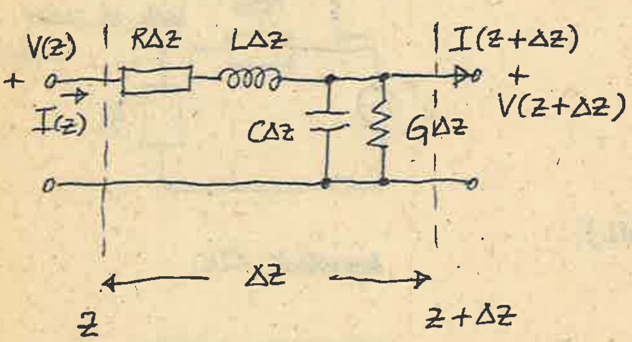
$$\Phi = \int_a^b \mu_0 H \cdot dr \, 1m^2 = \frac{\mu_0 I}{2\pi} \ln \frac{b}{a}$$

$$\frac{\Phi}{I} = \frac{\mu_0}{2\pi} \ln \frac{b}{a}$$

$$\frac{\mu_0}{8\pi} = \text{self inductance of inner conductor}$$

$$L \approx \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi} \ln \frac{b}{a}$$

Distributed Circuit Theory Approach



Equivalent cct. of a transmission line.

We will assume that the quantities are all phasor and the time dependence is $e^{\pm j\omega t}$

$$V(z+\Delta z) = V(z) - (R+j\omega L)\underbrace{I(z)}_{\text{Current}}\Delta z$$

$$\frac{V(z+\Delta z) - V(z)}{\Delta z} = -(R+j\omega L)\, I(z)$$

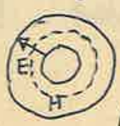
$$\lim_{\Delta z \to 0} \frac{V(z+\Delta z) - V(z)}{\Delta z} = \boxed{\frac{dV}{dz} = -(R+j\omega L)\, I(z)} \quad (1)$$

$$I(z+\Delta z) = I(z) - (G+j\omega C)\, V(z+\Delta z)\Delta z$$

$$\lim_{\Delta z \to 0} \frac{I(z+\Delta z) - I(z)}{\Delta z} = -(G+j\omega C)\, V(z)$$

$$\boxed{\frac{dI}{dz} = -(G+j\omega C)\, V(z)} \quad (2)$$

Note that on a transmission line, a voltage and a current can be defined only if there is a transversal electric and magnetic field in the transverse plane i.e. in the plane perpendicular to the transmission line.



TEM – mode
(Transversal electric – magnetic Mode)

Differentiating (1) and (2) w.r.t z and combining, we obtain:

(3) $$\frac{d^2V}{dz^2} = \gamma^2 V \qquad \gamma^2 = (R+j\omega L)(G+j\omega C)$$

(4) $$\frac{d^2I}{dz^2} = \gamma^2 I$$

The solutions of (3) and (4) are well known

$$V = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$I = I^+ e^{-\gamma z} - I^- e^{\gamma z}$$

It can be written concisely:

$$V = \left\{ e^{\pm \gamma z} \right\}$$

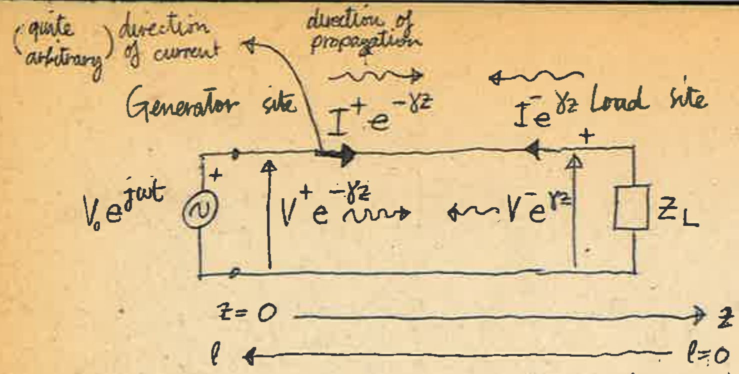
$$I = \left\{ e^{\pm \gamma z} \right\}$$

Then include time dependence : $e^{+j\omega t}$

$$\underbrace{V^+ e^{-\gamma z + j\omega t}} + V^- e^{+\gamma z + j\omega t}$$

$$V^+ e^{-\gamma z + j\omega t} = V^+ e^{-\alpha z} e^{-j\beta z + j\omega t} \quad (\gamma = \alpha + j\beta)$$

This is a voltage wave propagating along the line in the z-direction and which is attenuated.



$$V^- e^{\gamma z + j\omega t} = V^- e^{\alpha z} e^{j\beta z + j\omega t}$$

This is a voltage wave which propagates in $-z$ direction towards generator.
This is called the reflected voltage wave.

In practice usually we use l coordinates (Distance measured from load)

$$V = V^+ e^{\gamma l} + V^- e^{-\gamma l}$$

$$I = I^+ e^{\gamma l} - I^- e^{-\gamma l}$$

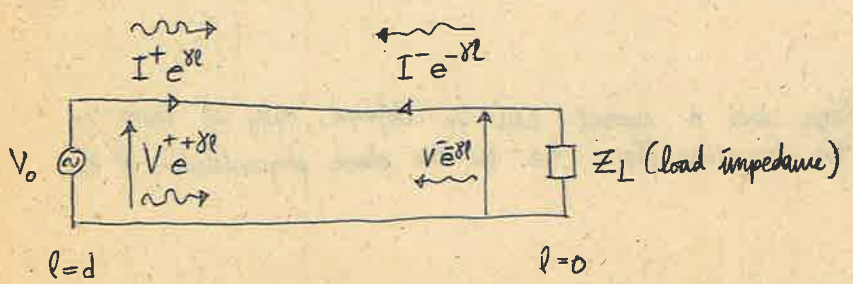
because directions are opposite.

$I^+ e^{\gamma l}$ $\rightarrow$

$\leftarrow$ $I^- e^{-\gamma l}$

$\rightarrow I_{TOT}$

09121980



total voltage $= V(l) = V^+ e^{\gamma l} + V^- e^{-\gamma l}$

total current $= I(l) = I^+ e^{\gamma l} - I^- e^{-\gamma l}$: from generator to load

$\cancel{\frac{dV^+}{dz} e}$

$$\frac{dV}{dz} = -(R + j\omega L) I$$

$$\frac{dI}{dz} = -(G + j\omega C) V \qquad z = d - l$$

$$\frac{dV}{dl} = (R + j\omega L) I \qquad (3)$$

$$\frac{dI}{dl} = (G + j\omega C) V \qquad (4)$$

Apply (3) to $\{V^+ e^{\gamma l}, I^+ e^{\gamma l}\}$

$$\frac{dV^+ e^{\gamma l}}{dl} = \gamma V^+ e^{\gamma l} = (R + j\omega L) I^+ e^{\gamma l}$$

So $\frac{V^+}{I^+} = \frac{R + j\omega L}{\gamma} = \frac{R + j\omega L}{\sqrt{(R + j\omega L)(G + j\omega C)}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = Z_0$ Characteristic impedance of transmission line **or** wave impedance for voltage and current waves.

Now apply (4) to $\{V^+ e^{\gamma l}, I^+ e^{\gamma l}\}$

$$\frac{dI^+ e^{\gamma l}}{dl} = (G + j\omega C) V^+ e^{\gamma l} = I^+ \gamma e^{\gamma l}$$

So $\frac{I^+}{V^+} = \frac{\gamma}{G + j\omega C} = \frac{\sqrt{(R + j\omega L)(G + j\omega C)}}{G + j\omega C}$

Apply (3) to $\left\{ V^- e^{-\gamma l}, -I^- e^{-\gamma l} \right\}$

$$\frac{d}{dl}\left(V^- e^{-\gamma l}\right) = (R+j\omega L)\left(-I^- e^{-\gamma l}\right)$$

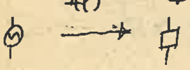
$$\frac{V^-}{I^-} = \frac{R+j\omega L}{\gamma} = Z_0$$

Similarly (4) gives $\dfrac{V^-}{I^-} = \dfrac{\gamma}{G+j\omega C} = Z_0$

We define $V(l)\Big|_{l=0} = V(0) = V_L$ as the load voltage

$I(l)\Big|_{l=0} = I(0) = I_L$ as the load current.

$$I(l) = I^+ e^{\gamma l} - I^- e^{-\gamma l} \qquad \frac{V(0)}{I(0)} = \frac{V_L}{I_L} = Z_L$$



$$V_L = V^+ + V^- = Z_0(I^+ + I^-)$$

$$I_L = I^+ - I^-$$

$$\frac{V_L}{I_L} = Z_L = Z_0 \frac{I^+ + I^-}{I^+ - I^-} = Z_0 \frac{V^+ + V^-}{V^+ - V^-}$$

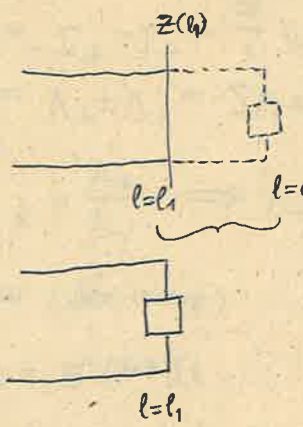
We will define a quantity $\Gamma_L = \dfrac{V^-}{V^+} = \dfrac{V^- e^{-\gamma l}}{V^+ e^{\gamma l}}\Bigg|_{l=0}$ load reflection coefficient

$$Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L}$$

or: $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

We define $Z(l) = \dfrac{V(l)}{I(l)}$ as the input impedance at l seen looking into the load.

Suppose that $l = l_1$, then for $l \geq l_1$



23 Ahmet Sezenoğlu

$$Z(l) = \frac{V^+ e^{\gamma l} + V^- e^{-\gamma l}}{I^+ e^{\gamma l} - I^- e^{-\gamma l}} = \frac{V^+ e^{\gamma l} + V^- e^{-\gamma l}}{\frac{1}{Z_0} V^+ e^{\gamma l} - \frac{1}{Z_0} V^- e^{-\gamma l}}$$

$$Z(l) = Z_0 \frac{1 + \dfrac{V^- e^{-\gamma l}}{V^+ e^{\gamma l}}}{1 - \dfrac{V^- e^{-\gamma l}}{V^+ e^{\gamma l}}} = Z_0 \frac{1+\Gamma(l)}{1-\Gamma(l)}$$

$\Gamma(l) = \dfrac{V^- e^{-\gamma l}}{V^+ e^{\gamma l}} = \dfrac{V^-}{V^+} e^{-2\gamma l} = \Gamma_L e^{-2\gamma l}$: Reflection Coefficient at any l

$(\Gamma(0) = \Gamma_L)$

Special load impedances :

(1) $Z_L = 0$ (Short circuit) $\qquad \Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

$$\Gamma_L = \frac{V^-}{V^+} = -1$$

$$V(0) = V_L = 0$$

$$I(0) = I_L = I^+ - I^- = \frac{V^+ - V^-}{Z_o} = \frac{2V^+}{Z_o} = 2I^+$$

$$\Gamma(l) = -e^{-2\gamma l}$$

$$Z_{in}(l) = Z_o \tanh \gamma l \qquad Z(l) = Z_{in}(l) = Z_o \frac{Z_L + Z_o \tanh \gamma l}{Z_o + Z_L \tanh \gamma l}$$

(2) $Z_L = \infty$ (open circuit)

$$\Gamma_L = 1 = \frac{V^-}{V^+} \Longrightarrow V^+ = V^-$$

$$V(0) = V^+ + V^- = 2V^+$$

$$I(0) = I^+ - I^- = \frac{1}{Z_o}(V^+ - V^-) = 0$$

$$Z_{in}(l) = Z_o \coth \gamma l = \cancel{\frac{Z_o \; 1}{1 + \tan}}$$

$$\boxed{[Z_{in}(l)]_{sh} \cdot [Z_{in}(l)]_{op} = Z_o^2}$$

(3) $Z_L = Z_o$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = 0 \quad \text{Then} \quad \begin{aligned} V^- &= \Gamma_L V^+ = 0 \\ I^- &= \Gamma_L I^+ = 0 \end{aligned}$$

There is no reflection MATCHED CASE

For practical transmission lines : $R \ll \omega L$, $G \ll \omega C$

Then we have low - loss lines $\quad \gamma = \alpha + j\beta$

$$\alpha \approx \frac{1}{2}\left(\frac{R}{Z_o} + G Z_o\right)$$

$$\beta \approx \omega\sqrt{LC}$$

$$Z_o \approx \sqrt{\frac{L}{C}}$$

For ideal ; (lossless) case

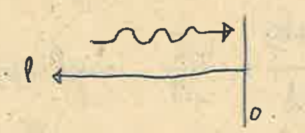
$$\beta = \omega\sqrt{LC}$$

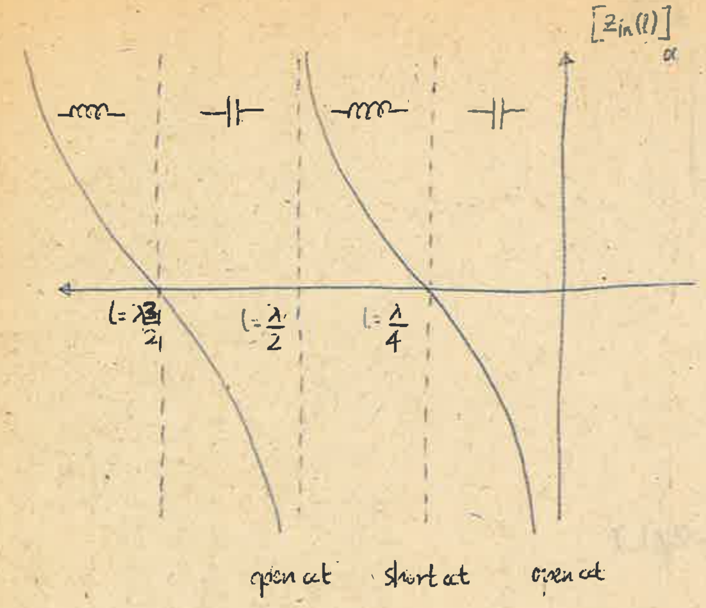
$$\alpha = 0$$

$$Z_o = \sqrt{\frac{L}{C}}$$

$\Gamma_l = \Gamma_L e^{-2\alpha l} e^{-j2\beta l}$ We will assume from now on that $\alpha l \ll 1$ for the practical length l.

$\boxed{\gamma = j\beta} \qquad V^+ e^{j\beta l + j\omega t}$





Total voltage and current : (Phasor diagram)

$$V(l) = V^+ e^{j\beta l} + V^- e^{-j\beta l} = V^+ e^{j\beta l}\left(1+\frac{V^-}{V^+}e^{-j2\beta l}\right)$$

$$I(l) = \frac{1}{Z_0}V^+ e^{j\beta l} - \frac{1}{Z_0}V^- e^{-j\beta l} = \frac{V^+}{Z_0}e^{j\beta l}\left(1-\frac{V^-}{V^+}e^{-j2\beta l}\right)$$

$$V(l) = V^+ e^{j\beta l}\,(AV) \quad \text{where } (AV) = 1 + |\Gamma_L|\, e^{j\theta'}$$

$$I(l) = \frac{V^+}{Z_0} e^{j\beta l}\,(AI) \quad \text{where } (AI) = 1 - |\Gamma_L|\, e^{j\theta'}$$

$$\theta' = \theta - 2\beta l$$

$$\Gamma_L = \frac{V^-}{V^+} = |\Gamma_L| e^{j\theta} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$Z_{in}(l) = \frac{V(l)}{I(l)} = Z_0 \frac{(AV)}{(AI)}$$

The phasor diagram :

$$\overrightarrow{AO} = 1.0$$

$$OV = |\Gamma_L| e^{j\theta'}$$

$$OI = |\Gamma_L| e^{j\theta' + j\pi} = -OV$$

$$\overrightarrow{AV} = |(AV)| = |1 + |\Gamma_L| e^{j\theta'}|$$

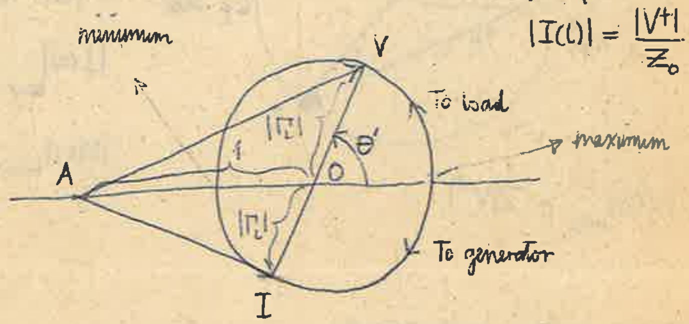
$$\overrightarrow{AI} = |(AI)| = |1 - |\Gamma_L| e^{j\theta'}|$$

So $\overrightarrow{AV} = \frac{|V(l)|}{|V^+|}$ ← magnitude of total voltage

$\overrightarrow{AI} = Z_0 \frac{|I(l)|}{|V^+|}$ ← magnitude of total current

$$|V(l)| = |V^+|\, \overrightarrow{AV}$$

$$|I(l)| = \frac{|V^+|}{Z_0}\, \overrightarrow{AI}$$



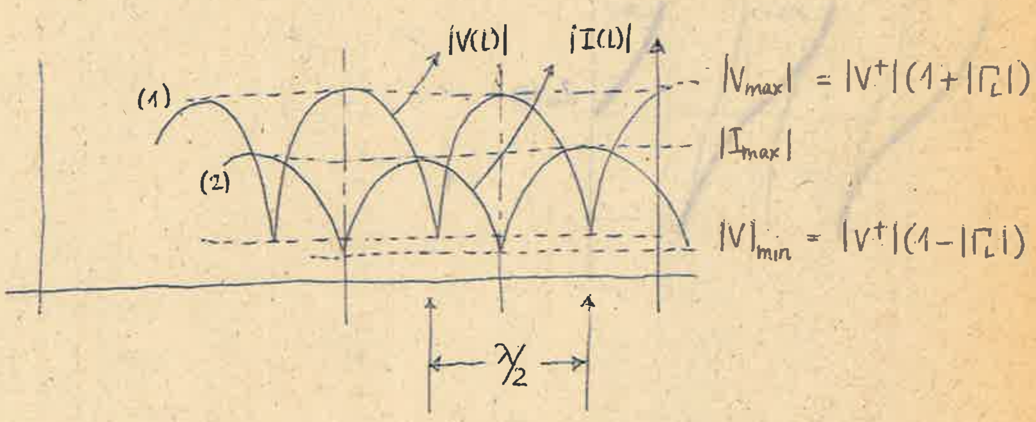
If you move on the line by a distance $\Delta l = \frac{\lambda}{2}$, total voltage, total current, input impedance do not change.

from diagram:

$$\overrightarrow{AI}_{max} = \overrightarrow{AV}_{max} = 1 + |\Gamma_L|$$

$$\overrightarrow{AI}_{min} = \overrightarrow{AV}_{min} = 1 - |\Gamma_L|$$

When total voltage is maximum, total current is minimum.



(1) VOLTAGE STANDING WAVE PATTERN

(2) CURRENT STANDING WAVE PATTERN

Show that

$$Z_{in}\left(l + n\frac{\lambda}{4}\right) \cdot Z_{in}(l) = Z_0^2$$

$n = 1, 3, 5 \ldots$

This is a $\frac{\lambda}{4}$ (quarter wave) transformer.

Voltage Standing Wave Ratio : $VSWR = \dfrac{|V(l_1)|_{max}}{|V(l_2)|_{min}} = \dfrac{\overrightarrow{A\,MAX}}{\overrightarrow{A\,MIN}}$

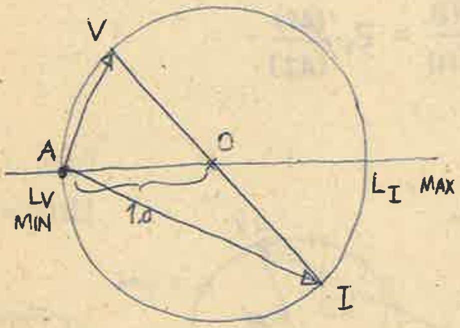
$$\boxed{S = VSWR = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}}$$

Exercise : a) let $l = \frac{\lambda}{4}$ Find the full relationship of $V(l)$, $I(l)$ to $V(0)$, $I(0)$

b) Repeat a for $l = \frac{\lambda}{2}$

Special Cases :

i) Short at ($Z_L = 0$, $\Gamma_L = -1$, $|\Gamma_L| = 1$, $\theta = \pi$, $\theta' = \pi - 2\beta l$)



L_V is the load point for voltage
L_I is the load point for current.

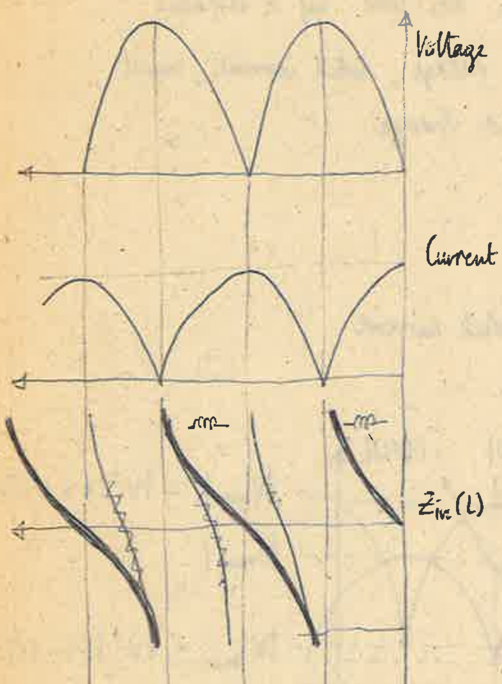
MIN is $1 - |\Gamma_L| = 1 - 1 = 0$

$$|I(0)|_{max} = \frac{|V^+|}{Z_0} \cdot 2$$

$$|V(0)|_{min} = 0$$

$$|V(l)|_{max} = 2|V^+|$$

Variation of voltage and current :



$$Z_{in}(l) = Z_0 \frac{AV}{AI} = Z_0 \frac{1 - e^{j(\pi - 2\beta l)}}{1 + e^{j(\pi - 2\beta l)}} = j Z_0 \tan\beta l$$

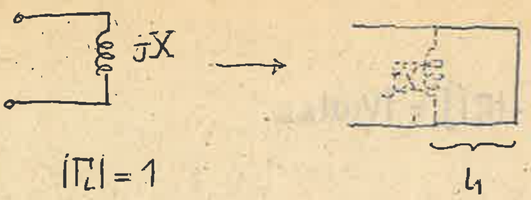
you can also simply use the formula

$$Z_{in}(l) = Z_0 \frac{Z_L + j Z_0 \tan\beta l}{Z_0 + j Z_L \tan\beta l} = j Z_0 \tan\beta l$$

Reactive Terminations

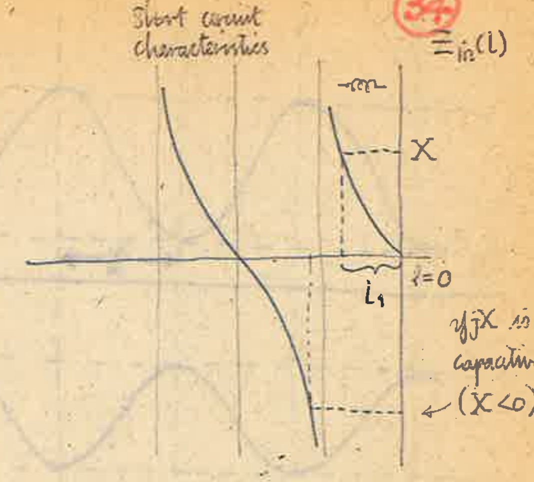
$Z_L = jX$

$X > 0$



$$\Gamma_L = \frac{jX - Z_0}{jX + Z_0} \qquad |\Gamma_L| = 1$$

$S = \infty$

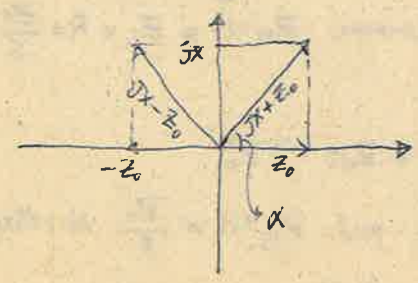


16121980

Finding the phase of Γ_L for inductive and capacitive terminations;

① Inductive:

$Z_L = jX \quad X \geq 0 \qquad \Gamma_L = \dfrac{jX - Z_0}{jX + Z_0} \qquad |\Gamma_L| = 1$



$$\Gamma_L = |\Gamma_L| e^{j\theta} = 1 e^{j\theta} = e^{j(\pi - 2\alpha)}$$

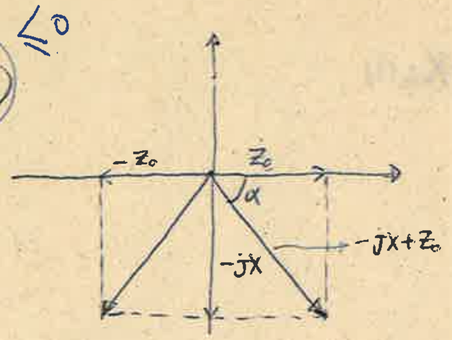
$$\alpha = \tan^{-1} \frac{X}{Z_0}$$

② Capacitive:

$\angle 0$

$Z_L = jX \;; \; (X \geq 0)$

$$\Gamma_L = \frac{-jX - Z_0}{-jX + Z_0}$$



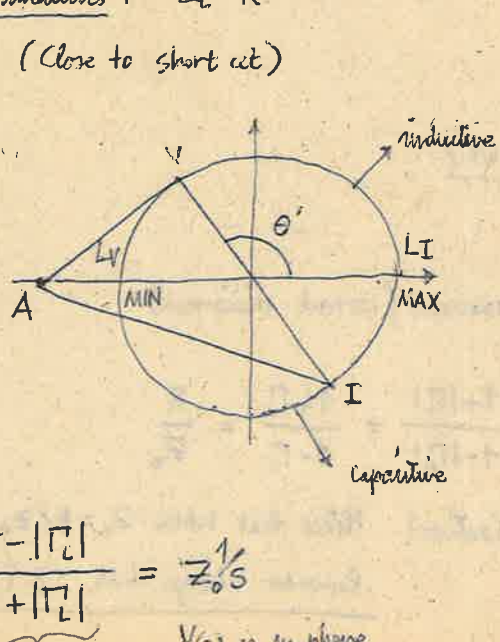
③ Purely resistive terminations: $Z_L = R$

Case 1: let $R < Z_0$ (Close to short cct)

$$\Gamma_L = \frac{R - Z_0}{R + Z_0} < 0$$

$$|\Gamma_L| = \frac{Z_0 - R}{R + Z_0}$$

$\theta = \pi$



$$|V(l)| = |V^+|\,|1 + |\Gamma_L| e^{j\theta'}|$$

$e^{j\beta l}$

$\theta' = \theta - 2\beta l$

$l = 0$ load point $\theta' = \theta = \pi$

$$|V(0)| = |V_L| = |V^+|\,|1 - |\Gamma_L||$$

$$|I(l)| = \frac{|V^+|}{Z_0} |1 - |\Gamma_L| e^{j\theta'}|$$

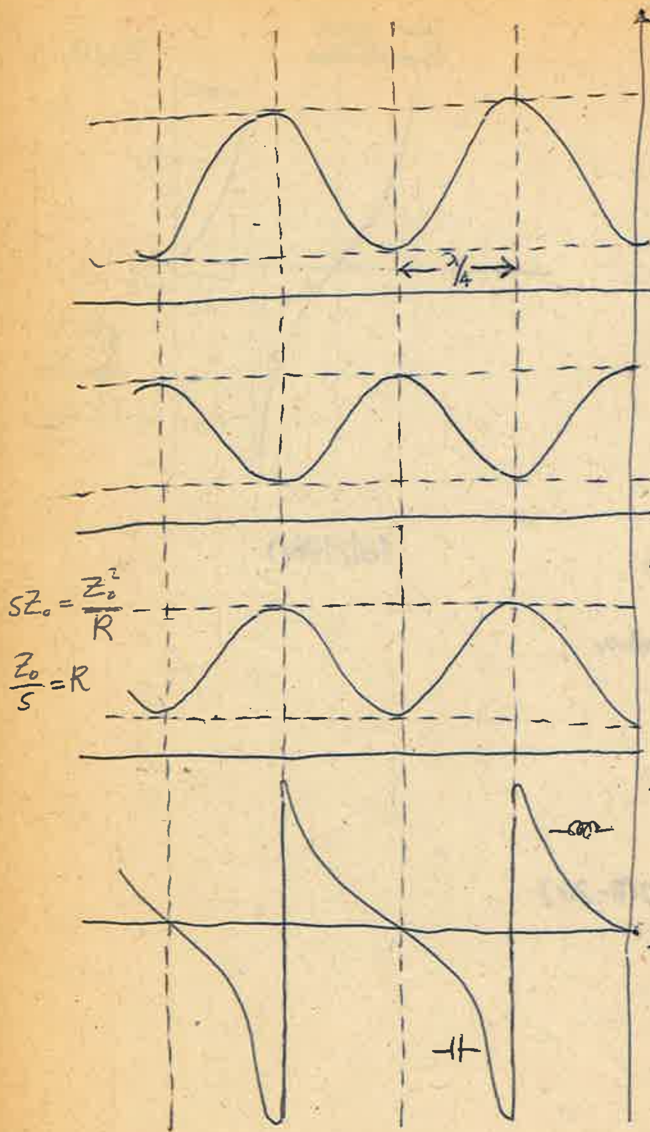
$e^{j\beta l}$

$$|I(0)| = \frac{|V^+|}{Z_0} |1 + |\Gamma_L||$$

$$\frac{|V(0)|}{|I(0)|} = Z_0 \underbrace{\frac{1 - |\Gamma_L|}{1 + |\Gamma_L|}}_{S = VSWR} = Z_0 \frac{1}{S}$$

$V(0)$ is in phase with $I(0)$ at the load.

PURELY RESISTIVE TERMINATION ($R < Z_o$)



$$|V^+|\left[1+|\Gamma_L|\right] = |V(l)|_{max}$$

$$|V^+|\left[1-|\Gamma_L|\right] = |V(l)|_{min}$$

$$\frac{|V^+|}{Z_o}\left[1+|\Gamma_L|\right] = |I(l)|_{max}$$

$$\frac{|V^+|}{Z_o}\left[1-|\Gamma_L|\right] = |I(l)|_{min}$$

at voltage minimum

$$Z_{in}(l) = Z_o \frac{1-|\Gamma_L|}{1+|\Gamma_L|} = \frac{Z_o}{s}$$ (At all voltage or current maximum points)

Notice that since $s \geq 1$ $Z_{in}(l) < Z_o$

and since also $R = Z_L = \frac{Z_o}{s}$

At all voltage minimum $Z_{in}(l) = Z_L = R = \frac{Z_o}{s}$

at voltage maximum :

$$Z_{in}(l) = Z_o \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = Z_o s > Z_o$$

since $R = \frac{Z_o}{s}$ $\quad s = \frac{Z_o}{R}$ and $Z_{in}(l) = \frac{Z_o^2}{R}$ at voltage max.

only for $R < Z_o$ $\quad = sZ_o$

$$Z_{in}(l) = Z_o \frac{R + jZ_o \tan\beta l}{Z_o + jR\beta l} = R_{in}(l) + jX_{in}(l)$$

[General formula]

You can show that :

$$R_{in}(l) = \frac{RZ_o^2}{R^2\sin^2\beta l + Z_o^2\cos^2\beta l}$$

$$X_{in}(l) = Z_o \frac{(Z_o^2 - R^2)\sin\beta l\cos\beta l}{R^2\sin^2\beta l + Z_o^2\cos^2\beta l}$$

for $Z_L = R$ (any value >0)

Case 2 PURELY RESISTIVE $R > Z_o$ (Similar to open cct)

$Z_L = R$

$\Gamma_L = \frac{R - Z_o}{R + Z_o} > 0$

$|\Gamma_L| = \Gamma_L$

$\theta = 0$

$\theta' = \theta - 2\beta l$

for $l = 0$ we determine the point L_V for which

$\theta' = \theta = 0$ so $L_V = MAX$

Results :

Load is voltage maximum (current minimum)

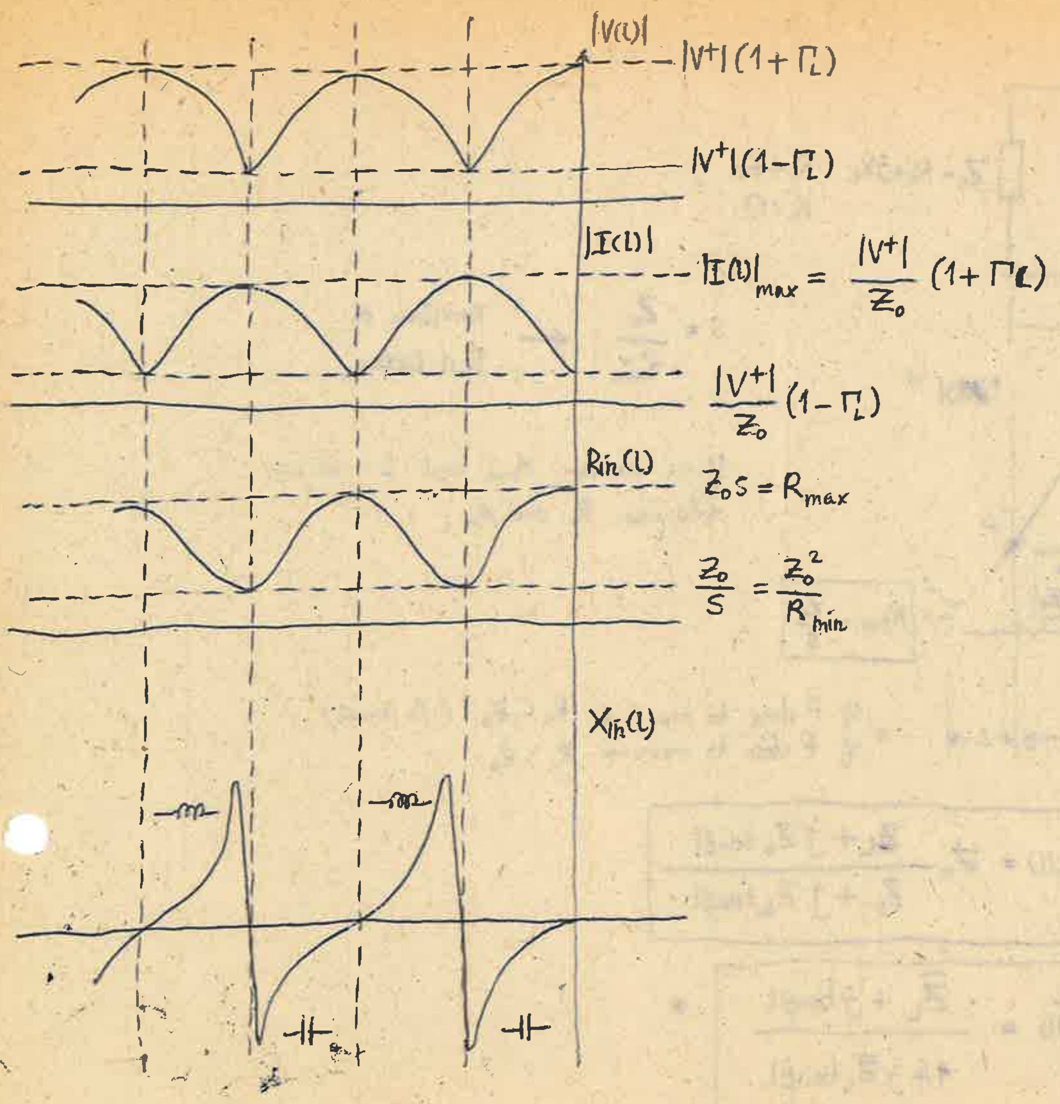
$Z_{in}(0) = R > Z_o$

$$s = \frac{|V(l)|_{max}}{|V(l)|_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+\Gamma_L}{1-\Gamma_L} = \frac{R}{Z_o}$$

$\boxed{s = \frac{R}{Z_o}}$ → Caution! Notice that when $Z_L = R < Z_o$ $s = \frac{Z_o}{R}$ in case 1

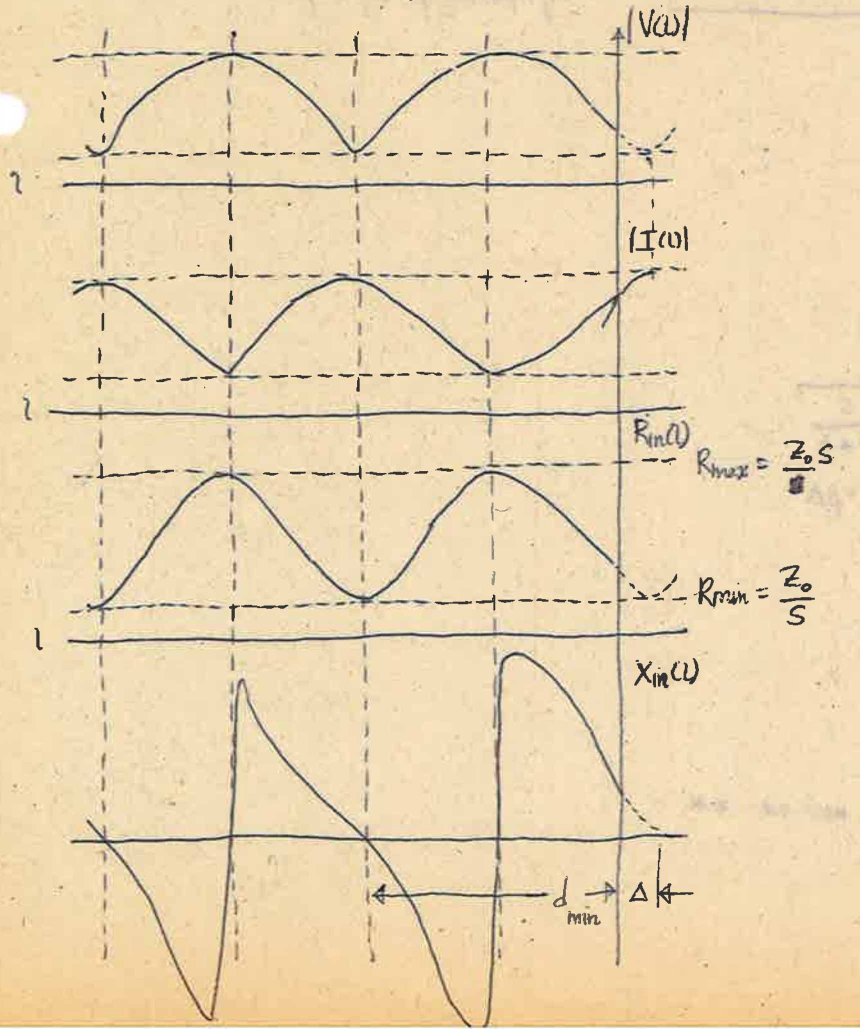
Remember always that $s \geq 1$!

FOR $R > Z_0 > 0$

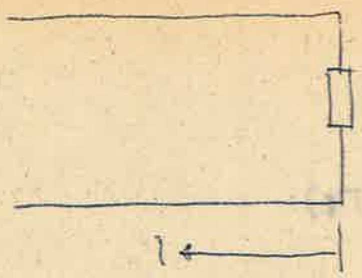


IMPEDANCE TERMINATION

ⓐ $Z_L = R + jX \quad R < Z_0, \; X \geq 0$ inductive



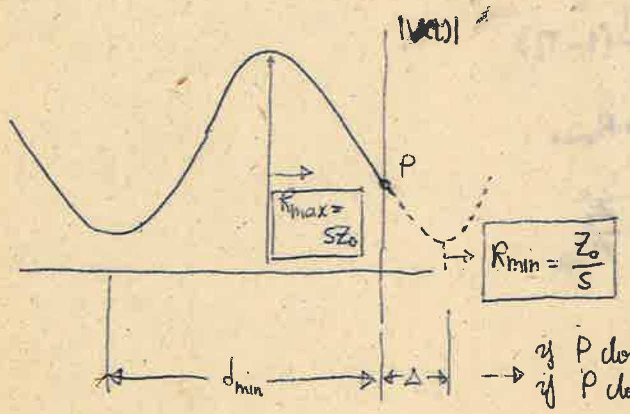
Example



$Z_L = R_L + jX_L \quad R_L < Z_0$

$X_L > 0$

$S = \dfrac{Z_0}{R_{min}}$ ← correction of last lecture.



If we measure d_{min} and S we can determine R_L and X_L

→ if P close to min → $R_L < Z_0$ (Δ small).
if P close to max → $R_L > Z_0$

$$\boxed{\overline{Z_{in}}(l) \triangleq \frac{Z_{in}(l)}{Z_0}} \qquad \boxed{Z_{in}(l) = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}}$$

$$\boxed{\overline{Z_L} \triangleq \frac{Z_L}{Z_0}} \qquad \boxed{\overline{Z_{in}}(l) = \frac{\overline{Z_L} + j\tan\beta l}{1 + j\overline{Z_L}\tan\beta l}} \quad *$$

Z_0 known

S, d_{min} measured

find : $\overline{Z_L}$

$$\boxed{\overline{Z_L} = \frac{\overline{Z_{in}}(l) - j\tan\beta l}{1 - j\overline{Z_{in}}(l)\tan\beta l}} \quad **$$

← may be obtained directly from * by replacing βl by $\pi - \beta l$

$Z_{in}(d_{min}) = R_{min} = \dfrac{Z_0}{S}$

$\overline{Z_{in}}(d_{min}) = \dfrac{1}{S}$

Also :

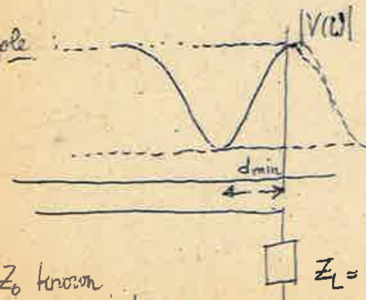
$$\overline{Z_L} = \frac{\overline{Z_{in}}(d_{min}) - j\tan\beta d_{min}}{1 - j\overline{Z_{in}}(d_{min})\tan\beta d_{min}}$$

Exercise Show that $R_L = Z_0$ when $\sin\beta\Delta = \sqrt{\dfrac{S}{1+j}}$

hint : use * set $\underbrace{Re\{Z_{in}(l)\}}_{=Z_0}$ and $\beta l = \beta\Delta$

calculate $\sin\beta\Delta$

Example



$Z_{in}(d_{min}) = \dfrac{Z_0}{S}$

$\overline{Z_{in}}(d_{min}) = \dfrac{1}{S}$ now use **

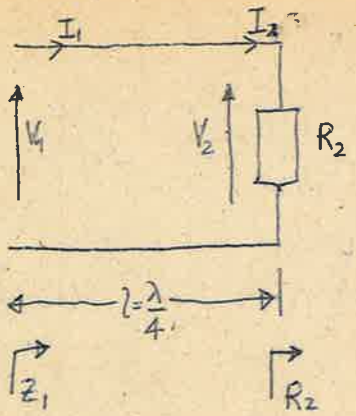
$Z_L = R_L + jX_L$?

Z_0 known

we measure S, d_{min}

we find that $d_{min} < \dfrac{\lambda}{4}$

Exercise: Consider:



V_1, I_1, V_2, I_2 are the total quantities

Show that: $V_1 = \frac{Z_0}{R_2} e^{j\frac{\pi}{2}} V_2$

$$I_1 = \frac{R_2}{Z_0} e^{j\frac{\pi}{2}} I_2$$

$$Z_1 = \frac{Z_0^2}{R_2}$$

Power Flow in Transmission lines

$$P^* = \frac{1}{2Z_0}(V^+ + V^-)(V^{+*} - V^{-*})$$

using $\Gamma_L = \frac{V^-}{V^+} = |\Gamma_L| e^{j\theta}$ we obtain:

$$P^* = \frac{|V^+|^2}{2Z_0}(1 - |\Gamma_L|^2) + j\frac{|V^+|^2}{2Z_0}|\Gamma_L| 2\sin\theta$$

$$P_{real,av} = Re\{P^*\} = \frac{|V^+|^2}{2Z_0}(1 - |\Gamma_L|^2) = \frac{|V^+|^2}{2Z_0}\frac{4s}{(s+1)^2}$$

$$P_{react,av} = Im\{P^*\} = \frac{|V^+|^2}{2Z_0}|\Gamma_L| 2\sin\theta = \frac{|V^+|^2}{2Z_0}\frac{s-1}{s+1} 2\sin\theta$$