The History Behind the Probability Theory and the Queuing Theory

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The author defended his DrPhilos thesis "Long-Term Telecommunications Forecasting" at NTNU, Trondheim ultimo 2006. His chosen lecture was: *"Are the odds against you? The history of how gambling initiated the theory of probability, and how the theory can be used to improve your odds"* (Stordahl, 2006). This article documents part of the lecture and extends the perspectives by drawing lines to establishment of the teletraffic/queuing theory.

Background

Humans have practised gambling at all times. The archaeologists have made excavations in prehistoric sites and found large numbers of roughly dice-shaped bones. Different types of games, sports events, other types of events and gambling are connected because it has always been challenging to make bets on different outcomes of a game.

Playing with dice, relief from ancient Rome

Experiences and simple statistics used more or less unconsciously made in old times the basis for the gamblers and their betting. Until the 16th century the mathematics was not applied on gambling and probability problems.

This paper shows how gambling problems initiated the mathematical theory of probability and gives an overview of the establishment of the mathematical theory of probability. The lines are drawn from the mathematical theory of probability to the establishment of the queuing/teletraffic theory more than 200 years later. And the pioneers in developing the queuing/ teletraffic theory were Agner Krarup Erlang and Tore Olaus Engset!

(Stordahl, 2006) identified that The Gambler's Ruin Problem was solved by using the same difference equations as for the M/M/1 queuing systems, only 200 years earlier. This paper investigates and explains the incitements for the development of the queuing/teletraffic theory which was mainly caused by introduction of telephone switching systems.

Gambling, caricature of an early roulette table, ca 1800

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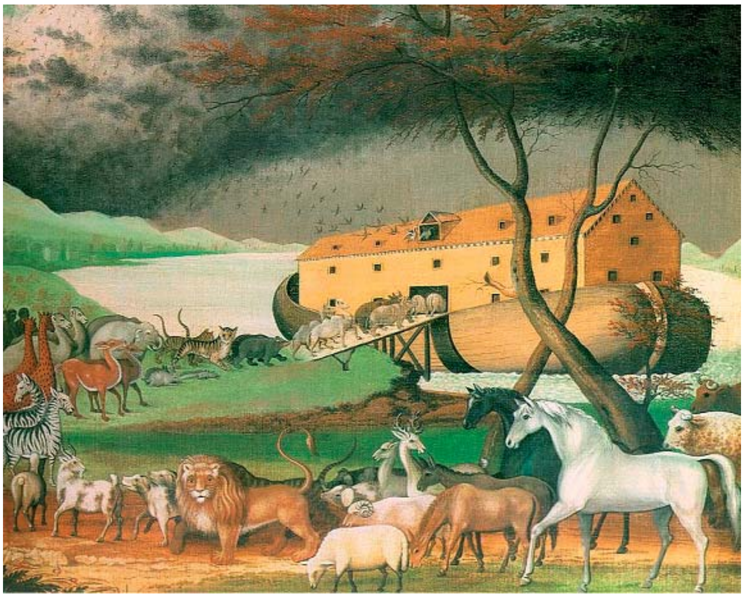


Early queuing system. Edvard Hicks (1780 – 1849): "Noah's Ark"

Before starting, a quotation from the famous mathematician Marquis Pierre-Simon de Laplace in his book *Thèorie Analytique des Probabilitiés* (Laplace, 1812) is in its place: *"It is remarkable that a science which began with consideration of games of chance should have become the most important object of human life ... The most important questions of life are, for the most part, really only problems of probability."*

The History Behind the Evolution of the Mathematical Theory of Probability

The most important sources used in this chapter are: (Todhunter, 1865), (Ore, 1953), (Ore, 1960), (King, 1990) and (Mahoney, 1994). The book by Isaac Todhunter from 1865, containing 1062 notes, is considered to be the real bible of the mathematical theory of probability. The book has on different subjects been shaded and supplemented by Anders Hall (Hall, 1990).

The very early start of the history of the mathematical theory of probability is strongly influenced by Gerolamo Cardano, Blaise Pascal, Pierre de Fermat and Christian Huygens.

Gerolamo Cardano (1501 – 1576)

Gerolamo Cardano is said to be the father of the mathematical theory of probability. Øystein Ore has written a very interesting book (Ore, 1953) where he analyses the many abilities of the unusual man and scientist Cardano. The last part of the book contains Cardano's book *Liber de Ludo Aleae (The Book on Games of Chance)* which Ore had translated from Latin.

Gerolamo Cardano's father, Fazio Cardano, was a lawyer, but he had excellent knowledge in medicine and mathematics. He was also consulted by Leonardo da Vinci regarding analysis in geometry.

The life of Gerolamo Cardano is very well described in his autobiography *De Propria Vita*. He was a well educated medical doctor. While studying he gambled to finance his studies. He also carried on gambling for many years. Cardano had many different interests and he published books in mathematics, physics, astronomy, probability theory, moral, upbringing, music, chess, dreams, death; but most of his books dealt with medical questions. In his autobiography he counted 131 printed works after having burned 170 manuscripts which he judged not to be good enough. Cardano's first book *On the Bad Practices of Medicine in Common Use* was published in 1536. The book was written because Cardano applied for but did not get a position at the hospital in Milan. However, after the publication the medical doctors were frightened and gave him a position. During the course of a few years he got the top position at the hospital. He was a skilful medical doctor and was often used by the aristocracy. He was also a unique debater who was impossible to beat in the duels held at that time.

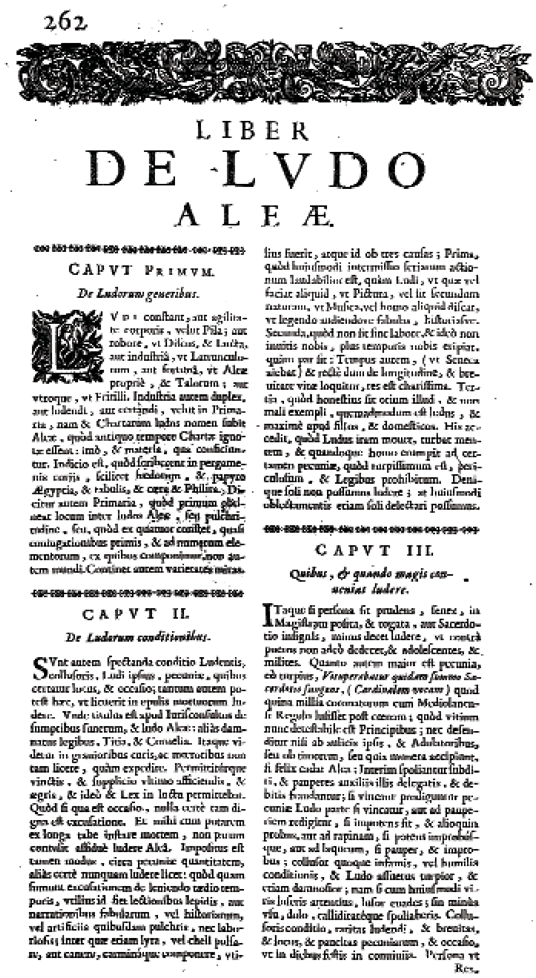
In 1545 Cardano issued *Ars Magna*, a textbook in arithmetic, where the solutions of the third and fourth degree equation were published for the first time. The solution of the third degree equation was originally found by Scipione del Ferro about 1500, but he did not publish his solution. At that time teachers at the universities in Italy could be challenged for their position through competitions where each duellist put up a set of questions to be solved by the other. Hence, it was better to have and to hide knowledge than to

Gerolamo Cardano (1501 – 1576) is said to be the father of the mathematical theory of probability share it. The solution of the fourth degree equation was found by Cardano's brilliant scholar Lodovico Ferrari.

Cardano wrote several books about gambling. The book *Liber de Ludo Aleae* is a handbook for gamblers. The book shows that Cardano was an experienced gambler. A lot of advice is presented about tricks and cheating, and recommendations are given to avoid cheating. Cardano applies basic principles for the probability theory. He defines probability as the number of favourable (outcomes) divided by the number of possible (outcomes). He states that the probability of a set of independent events is equal to the product of the probability of each of the events. He also touches the mathematical expectation and the law of large numbers. Cardano shows a complete sample space for throws with two and with three dice. His methods would in principle solve the Chevalier de Mère's problem, which, as will be seen was discussed by de Mère and Pascal about 100 years later.

The main part of the manuscript for the book was written at an early stage of Cardano's career, but some parts were included later. Unfortunately, he was not allowed to publish the book because in 1570 he was arrested by the Inquisition and denied further publishing. Regrettably, the book *Liber de Ludo Aleae* was not published until 1663, and then as a minor part of a large ten cover volume of Cardano's publications.

Therefore, *Liber de Ludo Aleae* did not make the impact on the evolution of the mathematical theory of probability as it could have done!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LIBER DE LVDO ALEÆ.

CAPVT PRIMVM.

De Ludorum generibus.

CAPVT II.

De Ludorum conditionibus.

CAPVT III.

Quibus, & quando magis conueniat ludere.

Gerolamo Cardano's book Liber de Ludo Aleae *was not published until 1663, then as part of* Opera Omnia*, a ten cover volume of Cardano's publications*

Blaise Pascal (1623 – 1662)

Blaise Pascal was taken very ill when he was one year old. Sickness followed Pascal through his life and he only lived to be 39 years old. When he was 16 he published a remarkable thesis on conic cuts. At the age of 18 he made the world's first calculation machine for addition and subtraction – in fact for helping his father with tax calculations, which was part of his job. He also carried out a series of pressure measurements and concluded on the existence of vacuum. Pascal was also an excellent author with a specific talent for polemics.

The literary style of Pascal was influenced by Antoine Gombaud Chevalier de Mère (1607 – 1684) who he got acquainted with in 1651/1652. Many books on the history of the mathematical theory of probability start with: *"A gambler named Chevalier de Mère presented two gambling problems to Blaise Pascal"*. (Ore, 1960) tells that "Chevalier de Mère would have turned in his grave at such a characterisa-

Blaise Pascal (1623 – 1662)

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The dice problem: How many times do you have to throw two dice to have a probability higher than 0.5 to get at least one double 6 in the sequence?

tion of his main occupation in life". Chevalier de Mère had received a good classical education and had experience from the army. He served in the court in Paris. He was a philosopher and a writer and he rapidly became a prominent figure at the court of Louis XIV where he was adviser in delicate situations and arbiter in conflicts.

Chevalier de Mère made Pascal aware of *The dice problems* which had been well known during the last centuries. One of the dice problems is described as follows: *How many times do you have to throw two dice to have probability higher than 0.5 to get at least one double 6 in the sequence?* Pascal solved the problem in the following way:

He stated that the probability not to get a double 6 in one throw is *35/36*. Then he postulated the same as Cardano did a hundred years earlier, that throws with dice are independent events and expressed that the probability not to get one double 6 in n throws is $(35/36)^n$. Hence, the probability to get at least a double 6 in n throws is $p_n = 1 - (35/36)^n$. Then, Pascal calculates $p_{24} = 0.491$ and $p_{25} = 0.506$. Hence, the limit is between throw 24 and throw 25.

At that time Pascal made contact with Fermat to get confirmation of his theories and this process is considered by many to be the start of the evolution of the mathematical theory of probability.

Pierre de Fermat (1601 – 1665), contemporary engraving

Pierre de Fermat (1601 – 1665)

Pierre de Fermat came from a wealthy merchant family on his father's side and a lawyer family on his mother's side. He studied law at the universities of Toulouse and Orléans and mathematics at the university of Bordeaux. He made a career as a lawyer and got continuously higher positions. It could be said that his professional career was as a lawyer, but he had mathematics as a lifelong hobby! Fermat was known for showing his mathematical results, but he did not always show his proofs. The reason was that he did not consider communicating his proofs to be his primary tasks in life. Significant parts of Fermat's scientific work have been found in the margins of manuscripts and in letters to his friends. In other words, he was not very interested in documentation and publication of his mathematical works. However, at a later stage of his life, in August 1654, he suggested to Carcavi that Carcavi and Pascal should publish his scientific work. His son made this possible 14 years after his death, in 1679.

In 2000, the 'World Mathematical Year', The Czech Republic issued a stamp showing Fermat's last theorem, also showing that Andrew Wiles proved it in 1995

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Fermat became known for making the foundation of the analytical geometry. He made significant contributions to the calculus through calculations of tangent, maximum and minimum and to the number theory. Fermat is also famous for his *last theorem* (however from 1637) where he states that there exist no integers which satisfy $x^n + y^n = z^n$ where $n > 2$. The proof is said to be made in the margin of one of his manuscripts.

Anyhow, the statement generated a fantastic story of how the best brains in mathematics over a period of more than 350 years tried to develop this proof (Singh, 1997). Then in 1993, based on seven years work in isolation, Professor Andrew Wiles from Princeton University, NJ, USA, publishes the proof at a mathematical conference in Cambridge. The event caused enormous publicity. However, a month later, it was shown that there was a 'hole' in the proof – it was incomplete. It was a catastrophe for Andrew Wiles.

He had isolated himself like the Pythagorean did 2000 years earlier when they developed mathematics as a religion – but only for the initiated! In spite of the enormous pressure from the media, Andrew Wiles succeeded one year later to complete the proof. In the completion he also used the Selmer groups. The Selmer groups were developed at the beginning of the 1950s by the Norwegian professor Ernst Selmer, who regrettably died on 8 November 2006.

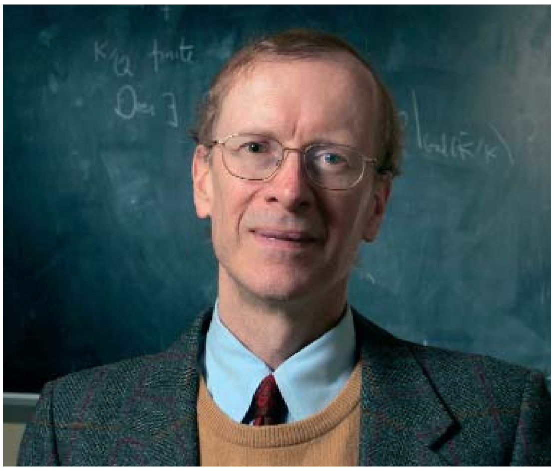
Start of the Mathematical Theory of Probability

The start of the mathematical theory of probability is by many considered to be the exchange of letters between Pascal and Fermat in 1654.

Pascal wanted at that time to get confirmation on some of his proofs on gambling problems. He made contact with the well known mathematician Roberval, but he did not get any support, only criticism. Roberval was described as "the greatest mathematician in Paris, and in conversation the most disagreeable man in the world". Pascal then consulted Fermat, who was living in Toulouse. Fermat was isolated from the mathematical environment and was happy to have contact with Pascal, which was reciprocated. There then followed an exchange of minimum seven letters, and this is by many considered as the real start of the mathematical theory of probability.

In the correspondence Fermat confirms Pascal's solutions on the dice problems.

However, the correspondence starts with the classical *Point problem,* which was well known from several centuries back. So far, nobody had found the solution.



Andrew Wiles, professor at Princeton University, NJ, USA, finally was able to prove Fermat's last theorem in 1995

There were different variants and wrappings of the problem. In the following is shown Pascal's solution to a simple Point problem.

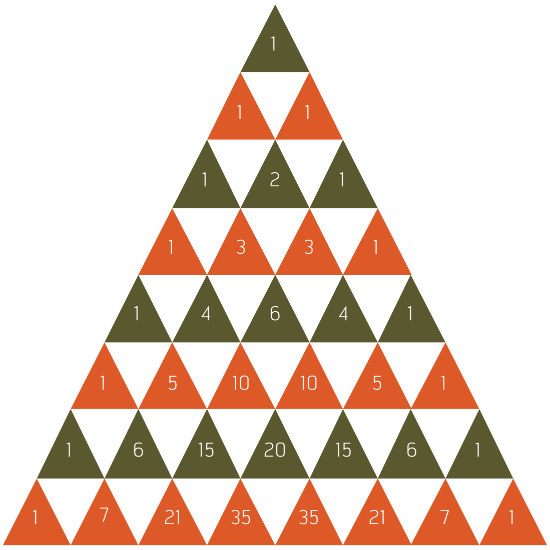
Two players A and B each put 32 gold coins (also called pistols) in the pot. The first player who gets three points has won the game and will get all the gold coins. The winner of each round gets one point. Each player has equal probability, 1:2, to win a round. However, the game is disrupted when player A has two points and player B one point. The question is: How should the pot be divided in a fair way?

Pascal allocates probabilities to different realisations and reasons as follows: The probability for player A to win the next round is 1:2. Because of that he should have 32 gold coins. The probability for player B to win the next round is 1:2. Then, player A and B both have two points and they should of course divide the remaining 32 gold coins equally, getting 16 gold coins each. Hence, player A should have a total of 48 gold coins and player B 16.

Fermat generalises the point problem and finds solutions for more complicated cases also when there are several players. He uses permutations, which can be used as long as the probability of winning is equal for each player. Fermat also finds some failures in Pascal's reasoning which he corrects.

The point problem. A fair division of gold coins (pistols)

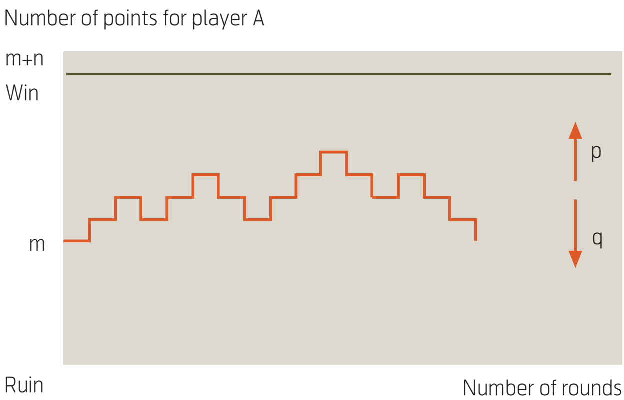
Then, Pascal develops solutions by using Pascal's triangle. Pascal's arithmetic



Pascal's triangle is a geometric arrangement of the binomial coefficients

triangle was well known, but got Pascal's name because he discovered new properties of the triangle which were previously not known. Pascal shows how binomial coefficients are used to calculate the probability of player A to win (and B) when A needs *m* points and B needs *n* points. Fermat also did the same using the Binomial distribution.

The exchange of letters stopped at the end of 1654 when Pascal sacrificed himself, as the rest of his family, for the Jansenism[1)]. His last years were spent mainly working with religious questions and in 1656 he wrote *Lettres Provinciales* opposing the Jesuits' attack on Jansenism.



The Gambler's Ruin problem

The Gambler's Ruin Problem

The correspondence between Pascal and Fermat was renaissanced a short period in 1656. Here, Pascal puts a question forward to Fermat which was known as the Gambler's Ruin problem. Fermat and Pascal solved the problem for some distinct values of the parameters, but it is not known that they solved the general problem.

The Gambler's Ruin problem is as follows: Player A starts with m points and player B with n points. The probability of player A winning one round is p, while the probability of player B winning a round is $q = 1 - p$. The winner of one round receives one point from the other player. The question is: What is the probability of player A winning the game; that is, to get $n + m$ points? And in general, what is the probability of player A winning the game when he has i points?

Christiaan Huygens (1629 – 1695) was next to improve the mathematical theory of probability significantly. In 1657 he published *Libellus De Ratiociniis in Ludo Aleae (The Value of all Chances in Games of Fortune)*, a book on probability theory. The book was published six years before Cardano's book and con-

Christiaan Huygens (1629 – 1695), engraving of Frederik Ottens, 18th century

1) *Jansenism was a branch of Catholic thought that emphasized original sin, human depravity, the necessity of divine grace, and predestination. Originating in the writings of the Flemish theologian Cornelius Otto Jansen, Jansenism formed a distinct movement within the Roman Catholic Church from the 16th to 18th centuries, but was condemned by the Roman Catholic Church as heretical (http://en.wikipedia.org/wiki/Jansenism)*

tains, among other things a precise definition of the concept of mathematical expectation. Huygens also puts up five unsolved probability problems; one of them, Huygens fifth problem, is The Gambler's Ruin problem which Huygens only solved for some values of the parameters.

Some more years passed before the general solution was found. In fact the solution was found by different approaches by James Bernouilli (1708), Montmort (1708), de Moivre (1712) and Struyck (1716), see (Hald, 1990). The process is a random walk with absorbing barriers in 0 and $n + m$.

A complete proof based on difference equations was first given by Struyck (1716), see (Hald, 1990, page 203). He finds the explicit solution of the difference equations first for $n = m$ and then for n different from m. Let $p(i)$ be the probability for player A to win the game given that he has i points. Then, the difference equations can be expressed by:

$$\begin{aligned} p(i) &= p\,p(i-1) + q\,p(i+1), \\ &\quad i = 1, 2, \ldots, m+n-1 \qquad (1) \\ p(0) &= 0 \\ p(m+n) &= 1 \end{aligned}$$

The first equation expresses that the probability of winning the game given that the player has i point is equal to p multiplied by the probability to win given that the player has i - 1 points pluss q multiplied by the probability that the player has i + 1 points. The two last equations are the edge conditions stating that the probability of player A winning is 0 if he has no points left (he is ruined) and the probability of winning when he has $n + m$ points is of course 1.

Now, we know that $p + q = 1$. The equation will not be changed when the left hand side is multiplied with $(p + q)$. Hence the equation is:

$$\begin{aligned} (p+q)\,p(i) &= p\,p(i-1) + q\,p(i+1), \\ &\quad i = 1, 2, \ldots, m+n-1 \qquad (2) \end{aligned}$$

This difference equation is exactly the same as the difference equation which describes the M/M/1 queuing system in statistical equilibrium. *However, the development of the queuing theory was not in place until 200 years later. The history of the evolution and the incitements for the evolution is treated in the last part of the paper.*

Queuing Models and Queuing Theory

Queuing theory is the mathematical study of waiting lines or queues. The theory enables mathematical analysis of several related processes, including arriving at the queue, waiting in the queue, and being served by the server(s) at the front of the queue. The theory permits the derivation and calculation of several performance measures including the average waiting time in the queue or the system, the expected number waiting or receiving service and the probability of encountering the system in certain states, such as empty, full, having an available server or having to wait a certain time to be served.

The simplest form of queuing models are based on the birth and death process, where the birth process describes the inter-arrival time (time between two arrivals) to the queue and the death process describes the service or holding time in the queue.

For queuing theory, it has been found convenient, if possible, to work with probability distributions which exhibit the memorylessness property, as this commonly simplifies the mathematics involved.

The memorylessness property is often denoted a Markovian property and a process with a Markovian property is called a Markov process, which means that the probability distribution of future states of the process, given the present state and all past states, depends only upon the present state and not on any past states.

Andrej Markov (1856 – 1922)

As a result, queuing models are frequently modeled as Poisson processes through the use of the exponential distribution.

Birth-death processes have many applications in demography, queuing theory, and in biology, for example to study the evolution of bacteria. The state, i, of the process represents the current size of the population. The transitions are limited to births and deaths. When a birth occurs, the process goes from state i to $i + 1$. When a death occurs, the process goes from state i to state $i - 1$.

The Markov process is named after the famous Russian mathematician Andrej Markov (1856 – 1922). Andrej Markov was created honorary doctor at the University of Oslo at the Abel jubilee in 1902 (Nils Henrik Abel (1802 – 1829))

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The Poisson process, discovered by the French mathematician *Siméon-Denis Poisson* (1781 – 1840) is a pure-birth process, the simplest example of a birth-death process. The Poisson distribution is a discrete probability distribution. It expresses the probability, $p(i, \lambda)$, of a number of events i, occurring in a fixed period of time, if these events occur with a known average rate λ, and are independent of the time since the last event.

$$p(i, \lambda) = \frac{e^{-\lambda}\lambda^i}{i!}$$

Suppose that the inter-arrival time is described by an exponential distribution with parameter λ (traffic intensity), and the holding time is described by an exponential distribution with parameter μ. Then the transient behavior of the queuing system is expressed by:

$$p_i'(t) = \lambda\, p_{i-1}(t) + (\lambda + \mu) p_i(t) + \mu\, p_{i+1}(t) \quad (3)$$

where $p_i'(t)$ is a derivative to $p_i(t)$ which is the probability to have i in the queue system at time t. The system is described as a function of time and can be solved when we know the starting value at time 0.

Suppose that the system reaches statistical equilibrium. Then the solution is independent of the starting values. In addition, it has a balance between inter-arrivals and services which implies $\lambda/\mu < 1$. Then $p_i'(t) = 0$.

Letting $p_i(t) = p_i$, we get:

$$(\lambda + \mu)p_i = \lambda p_{i-1} + \mu p_{i+1} \quad (4)$$

which is identical to the Gambler's Ruin problem equation (2), but with different notations.

This queuing system is denoted M/M/1: Exponential inter-arrival time and holding time and one server. The classification of queuing systems follows Kendell's definition (Kendell, 1953). The solution is found by expressing all the $\{p_i\}$ as a function of p_0 and then normalize based on the summing up of all the probabilities to 1. The same procedure is done for the Gambler's Ruin problem, but the edge conditions are also taken into account.

To show the equality in the solutions the following notations are used:

$$p(i) = p_i \quad (5)$$

$$\rho = \lambda/\mu = p/q \quad (6)$$

$$K = n + m \quad (7)$$

Solutions of different queuing systems and the Gambler's Ruin problem:

The Gambler's Ruin problem:

$$p(i) = (1 - \rho^i)/(1 - \rho^K) \quad (8)$$

M/M/1: $$p(i) = (1 - \rho)\, \rho^i \quad (9)$$

M/M/1/K: $$p(i) = \rho^i\, (1 - \rho)/(1 - \rho^{K+1}) \quad (10)$$

M/M/∞: $$p(i) = (\rho^i / i!)\, e^{-\rho} \quad (11)$$

M/M/K/K:

$$p(i) = (\rho^i/i!)/\left(\sum_{k=1}^{K} \rho^k/k!\right) \quad (12)$$

Erlang's B loss formula:

$$p(K) = (\rho^K/K!)/\left(\sum_{k=1}^{K} \rho^k/k!\right) \quad (13)$$

Here, A/B/C/D follows the notation of (Kendell, 1953) and (Kleinrock, 1975) where:

A: Interarrival time distribution
B: Service time distribution
C: Number of servers
D: Waiting room capacity

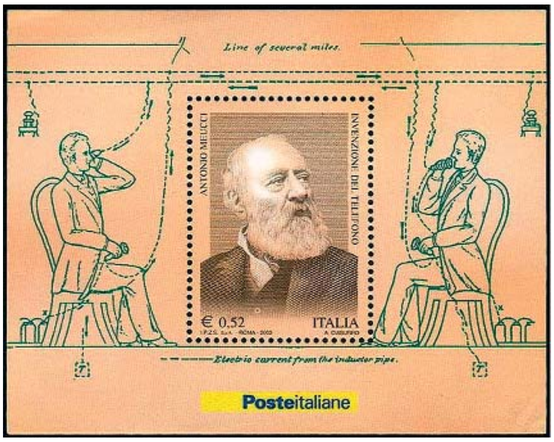
It could be noted that the solution of queuing models with more than one server uses $(i+1)\mu$ instead of μ in equation (4).

It is interesting to note that most books in queuing theory use equation (4) as a standard formula because it is derived from the transient equation (3). The stationary state equations can be interpreted as follows: The traffic stream out of state i is equal to the traffic stream into state i. Looking at the original work of Erlang (Erlang 1917) (Brockmeyer, 1948) he uses another approach. Instead of assuming that the traffic stream out of a state is equal to the traffic stream into the state, he postulates that the traffic stream is equal both ways between a cut of states. He then gets the simplified equation:

$$\lambda\, p(i) = (i + 1)\mu\, p(i + 1) \quad (14)$$

where he also uses $(i + 1)\mu$ as a more general expression. Arne Jensen uses the same approach in his paper (Jensen, 1954). A complete proof is given in (Morris, 1961). The possibility of using the cut is a much more powerful approach for modelling more complicated stationary queuing systems. This is shown in (Stordahl, 1972).

Let us go back and look at the real incentives for developing the queuing and teletraffic theory. The whole thing started with the telephone!



Antonio Meucci (1808 – 1889), the real inventor of the telephone was portrayed on an Italian stamp in 2005

The Telephone and the Manual Switches

The Italian Antonio Meucci (1808 – 1889), who had already created the first model of a telephone in Italy in 1834, tested electric transmission of the human voice in Cuba in 1849 and demonstrated his electric telephone in New York, USA in 1850. He had paid for a 'caveat' for the telephone in 1871. In the summer of 1872 Meucci asked Edward B. Grant (Vice President of American District Telegraph Co. of New York) permission to test his telephone apparatus on the company's telegraph lines. He gave Grant a description of his prototype and copy of his caveat. Up to 1874 Meucci only had enough money to renew his caveat while looking for funding for a true patent. After waiting two years without receiving an answer, Meucci went to Grant and asked him to be given back his documents, but Grant answered that he had lost them. The same year the caveat expired because Meucci lacked the money to renew it.

> *In 2002 the American Congress announced that Antonio Meucci (not Alexander Graham Bell) was the real inventor of the telephone.*
> (Kunnskapsforlaget, 2007)

Meanwhile, in 1867 the following could be read in an American newspaper: "The 46 years old, Joshua Coppersmith has been arrested in New York for attempting to extort funds from ignorant and superstitious people by exhibiting a device which he says will convey the human voice over metallic wires, so that it will be heard by the listener at the other end. He calls the instrument a *telephone*, which is obviously intended to imitate the word 'telegraph', and win the confidence of those who know of the success of the latter instrument without understanding the principles on which it is based. Well-informed people know that it is impossible to transmit the human voice over wires as may be done with dots and dashes and signals of the Morse Code, and that were it possible to do so, the thing would be of no practical value. The authorities who apprehended this criminal are to be congratulated, and it is to be hoped that it may serve as an example to other conscienceless schemers who enrich themselves at the expense of their fellow creatures."

Eight years later Alexander Graham Bell and his assistant Thomas Watson started to work on a device they called *A musical telegraph*, and on 10 March 1876 they succeeded in completing the device, which was eventually named the telephone. Bell applied for a patent for the invention and got it, but it was a close race since another American, Elisha Gray, applied for patent of a similar device *only two hours later!*

The telephone was demonstrated at the World exhibition in Philadelphia in May 1876. In one of the juries at the exhibition was a Norwegian, Joak Andersen,

Alexander Graham Bell, the later inventor of the telephone (1847-1922). The picture shows a well-known scene were Bell speaks on the phone between New York and Chicago in 1892. (Gilbert H. Grosvenor Collection, Prints and Photographs Division, Library of Congress)

Alexander Graham Bell (1847 – 1922) was a scientist and innovator. Born and bred in Scotland, he emigrated to Canada in 1870, and the following year to the United States. Bell is widely acclaimed for developing and patenting the telephone (at the same time but independently from Elisha Gray, and with prior efforts from Antonio Meucci and Philipp Reis). In addition to Bell's work in telecommunications, he was responsible for important advances in aviation and hydrofoil technology.

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Lars Magnus Ericsson (1846 – 1926), the founder of L.M. Ericsson

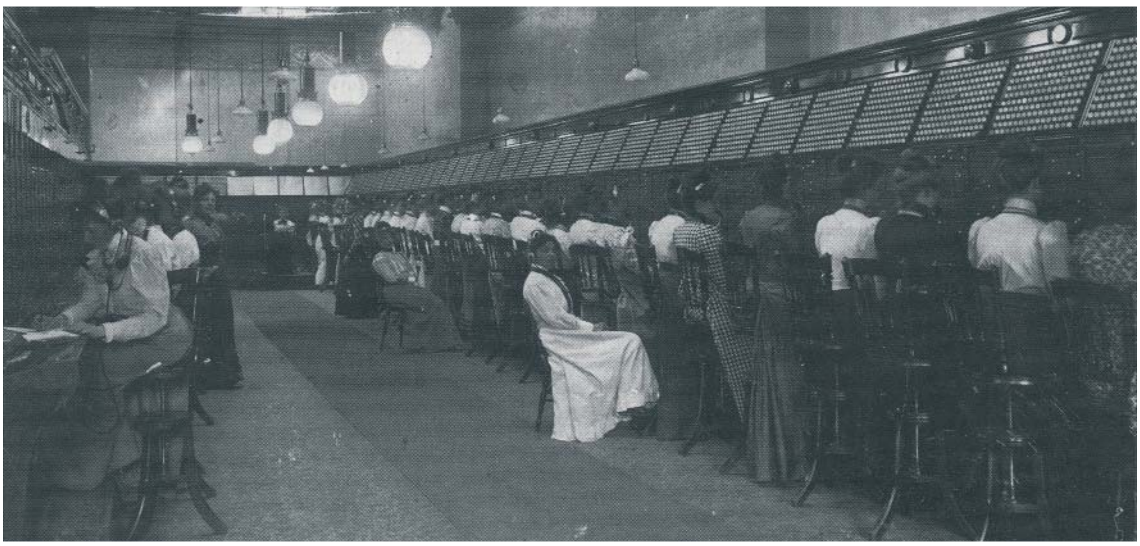
vice-consul to Denmark, who got two telephones which he sent to his son in Ålesund. However, the first public demonstration of the telephone in Norway was done in Bergen on 22 July 1877. The painter Johan Eimrich Rein had received two telephones from a friend who got them from Alexander Graham Bell. Then the engineer Jens Hopstock started a tour of Norway where he demonstrated the new invention. He also demonstrated the telephone in Stockholm and was even invited to King Oscar II to demonstrate it. Jens Hopstock was later appointed The International Bell Telephone Company's representative for Scandinavia (Bestorp, 1990).

Already in the autumn 1877 imitations of the telephone were made by Siemens & Halske in Germany. This prototype was the inspiration to the young instrument maker *Lars Magnus Ericsson* who started production of telephones later that year. He also founded the company L.M. Ericsson.

The problem up till now was the one-to-one telephone line correspondence between subscribers. Therefore, the next important step was the development of the manual switching system to reduce the size of the mesh network. The first manual switching system was opened on 28 January 1878 in New Haven, Connecticut. The same year manual switches were installed in London and Paris. The International Bell Telephone Company installed the first manual switch in Kristiania (the former name of Oslo), Norway in June 1880.

The International Bell Telephone Company was established by Bell's father-in-law, Gardiner Hubbard in 1879. The company installed switches and access networks in several large cities in Europe. The European headquarters was in Antwerp where the company in cooperation with Western Electric built a large factory for telephone equipment. The company got a strong position, especially in Belgium, The Netherlands and Russia. The company charged their customers heavily and *prevented a natural evolution of the telephone penetration in these countries*.

In Kristiania, the Bell Company started by charging 100 NOK per year for a subscription, which corresponded to a two month salary for a telephone operator. The price was raised continuously and in the spring 1881 the subscription price was 200 NOK. The company received a lot of criticism. The prices were too high, the company did not cooperate sufficiently with the authorities regarding the telephone line tracks, and the building owners complained about the installers. The Government in Kristiania, as opposed to several other large European cities, had not given The International Bell Telephone Company sole rights for the telephone system. Therefore, Carl Söderberg and 12 businessmen from Kristiania founded *Christiania Telefonforening* on 24 May 1881. Söderberg had the L.M. Ericcson agency for



Manual telephone exchange (Bestorp, 1990) telephone equipment in Norway, but telephone devices also from other telephone manufacturers were available. Carl Söderberg established in 1882 the independent Norwegian company, Elektrisk Bureau (EB); a competitor to L.M. Ericsson, which some years later had a yearly production capacity of 25,000 telephone sets and a considerable export.

Christiania Telefonforening's annual subscription price was set to 40 NOK, with 220 NOK for the installation. The Bell Company had to respond and reduced their subscription fee to 125 NOK, then to 100 NOK and later to 50 NOK.

The two competitors fought very hard to capture market share. The consequence was increased telephone penetration. The Bell Company did not succeed in getting sole rights in other Norwegian cities (Bestorp, 1990), (Rinde, 2005).

In 1885, the number of telephone subscriptions was 995 from Bell and 634 from Christiania Telefonforening. 230 subscribers had subscriptions in both networks. At that time the city government in Kristiania made it clear that the two companies had to merge because of the mess of telephone lines 'everywhere' and the possibilities for rationalisation and coordination.

The companies also got an ultimatum that they were not allowed to expand until a merger had taken place. The new company was the private stock company, *Christiania Telefonselskap*, which was established on 1 January 1886. A telephone monopoly was then established in Kristiania. However, the telephone subscription prices were on a reasonable level and the prices stayed constant for many years (Bestorp, 1990).

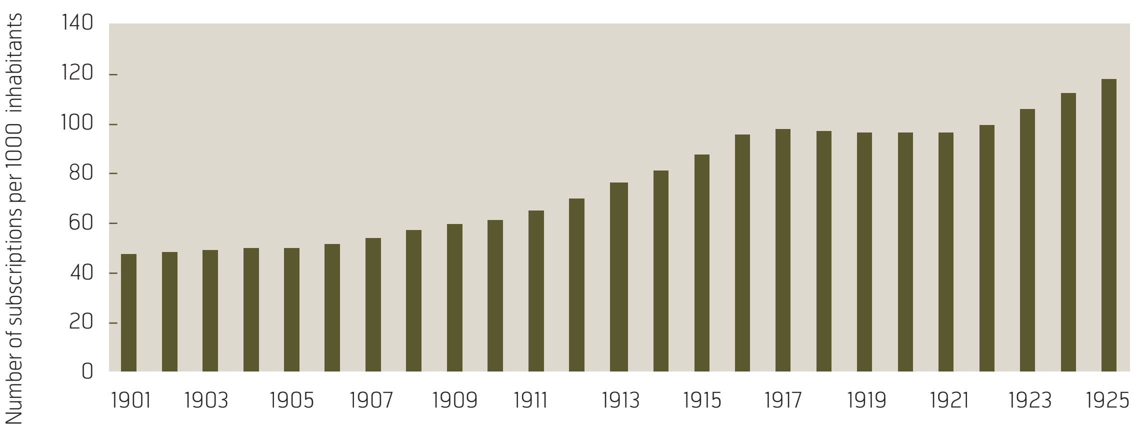
The International Bell Telephone Company, which originally was a threat to the telephone availability in Kristiania, had generated hard competition, low telephone prices and high demand. What happened in Kristiania also influenced the evolution in other places in Norway.

Sweden	15.6
Norway	15.0
Switzerland	12.4
Denmark	11.0
Germany	5.1
UK	5.1
Netherlands	3.3
Belgium	2.6
France	1.8
Austria	1.2
Spain	1.0
Hungary	0.9
Romania	0.3
Russia	0.3

Telephone penetration per 1000 inhabitants in European countries in 1900 (Rinde, 2005)

The table gives an overview of the telephone penetration in Europe in year 1900. The table shows that the Scandinavian countries together with Switzerland had the highest penetration. The penetration in Norway was 4.5 times higher than in The Netherlands and 5.8 times higher than in Belgium and about 3 times higher than the penetration in the well developed countries Germany and the UK.

The very special telephone growth in the Scandinavian countries during the first years could be called the *Scandinavian wonder* (Christensen, 2006). This observation is strengthened through the later mobile and broadband evolution in the Nordic countries. The Nordic countries have been pioneers in introducing



The evolution of the telephone penetration in Kristiania 1901 – 1925 (Bestorp, 1990) the Nordic Mobile System, NMT, already in the early 1980s, many years before other countries got the service. At the end of 2006 all five Nordic countries are among the eight OECD countries with the highest broadband penetration. As a conclusion all Nordic countries start very early to adopt new telecommunication technology.

In 1900 the number of subscribers in Kristiania was 9,864. There were 11,503 telephone sets, and each subscriber made in average 10.5 calls per working day. *A total of 27.8 million calls were carried through the main switch in Kristiania that year.* The telephone traffic was considerable (Bestorp, 1990).

Heavy investments in the national long distance network because of the increased traffic cleared the way for the start of a national telecommunication monopoly. The minister in charge of telecom, Jørgen Løvland, argued for the monopoly and in 1899 a law was passed giving Telegrafverket the exclusive rights to run telecom networks in Norway (Rinde, 2005), (Christensen, 2006).

Telegrafverket took over Christiania Telefonselskap on 1 January 1901. However, it turned out to be very expensive for the state to buy all the private telephone companies. As late as in the mid 1970s all private telephone companies in Norway were bought and embodied in Televerket (Telenor).

The first automatic telephone exchange in Norway was installed and operational in Skien in 1921 (Telektronikk, 61 (1-2), 1965, p 17)

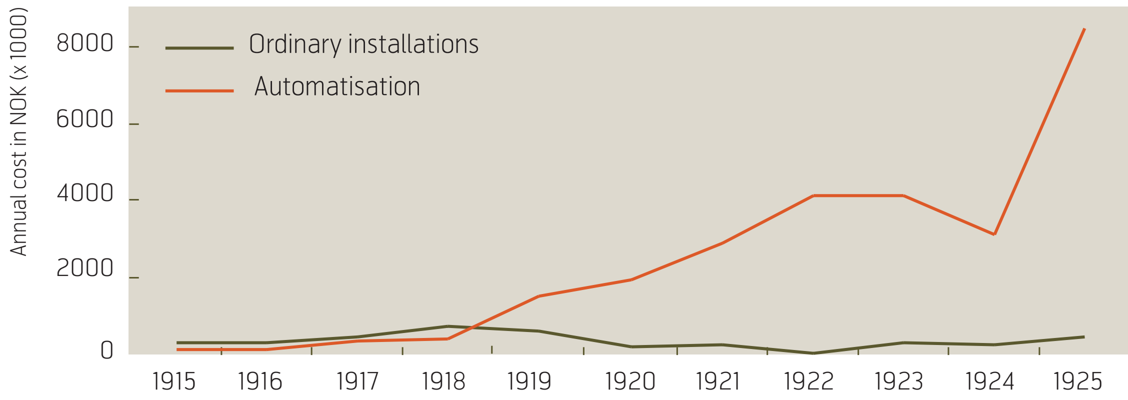
The subscription growth in Kristiania stagnated in the first years of the new century with only a few hundred new subscriptions each year. From 1907 the growth increased again and the total number of telephone subscriptions reached about 22,000 in 1913. During this period the government did not release sufficient investment means and problems with the traffic started to occur.

Start of the Automation

In 1914 only few subscribers could be connected to the telephone system in Kristiania. The situation was predicted several years earlier by Telegrafverket. The traffic increased and subscriber lines had to be moved from one switching group to another. Bestorp gives a detailed description of the situation in Kristiania caused by the increased traffic (Bestorp, 1990). A lot of subscriber complaints were received and the flood of complaints started in 1910. The subscribers were also irritated because the government used Telegrafverket as a money machine without giving back necessary investment means. The situation grew worse over the next years. From time to time parts of the network were totally blocked because of heavy traffic. In 1914 the Norwegian Parliament decided to increase the investments, but later lack of available equipment because of World War I limited the possibilities for expansion of the access network. The Ministry of Trade in cooperation with Telegrafverket appointed in 1918 a committee for withdrawing subscriptions from the subscribers. During a two year period 1,100 telephone sets were withdrawn in Kristiania and in the autumn of 1920 there were about 6,000 people on the waiting list. Copenhagen had about 5,000 people on the waiting list for telephone subscription.

This situation with considerable traffic problems and waiting lists for subscriptions was the backdrop for the pioneer work of Tore Olaus Engset and Agner Krarup Erlang. The development of traffic models for dimensioning of the switches and the access network was extremely important in a situation without sufficient investment means or available equipment.

The manual switches also had limitations. When the number of subscribers and traffic per subscriber increased, the capacity, including the number of telephone operators had to be increased. For a period the physical limits of the telephone company's premises prevented further expansion. And in July 1918 the Spanish flu hit Kristiania. Many telephone operators



Annual expenses for ordinary installations and automation in Kristiania 1915 – 1925 (Bestorp, 1990)

became infected causing significant traffic problems because of the lack of 'womanpower' at the switches.

In parallel, work was being done to start ordering new telephone switches. A three man committee with Telephone Director Iversen, Head of Department Engset, and Chief Engineer Abild was appointed in 1910. The committee's mandate was to give recommendations to the choice of a future switching system for Kristiania, either manual, semi-automatic or fully automatic. They spent 48 days travelling around Europe and 71 in the United States in 1911/1912. The recommendation was finalised in 1913, proposing fully automatic switching systems with primary and secondary exchanges. This envisaged a plan for 30,000 lines with a potential for 90,000 lines for Frogner exchange in Kristiania. The Norwegian Parliament sanctioned the plan in 1916 and Western Electric got the contract the same year. The first automatic exchange in Norway was to have been installed in 1917, but the ship *Kristianiafjord* transporting the exchange sank in June 1917. Because of World War I the project was delayed and the exchange was finally installed in 1921. At that time the private telephone companies in Skien and Bergen had already installed automatic exchanges, while a city like Stockholm still only had manual switches.

The figure above shows that investments for the establishment of automatic exchanges were very high in the first part of the 1920s. On the other hand, the investments also caused a significant reduction in the number of telephone operators. The number was reduced from 610 persons to 347 persons during the period 1924 – 1925.

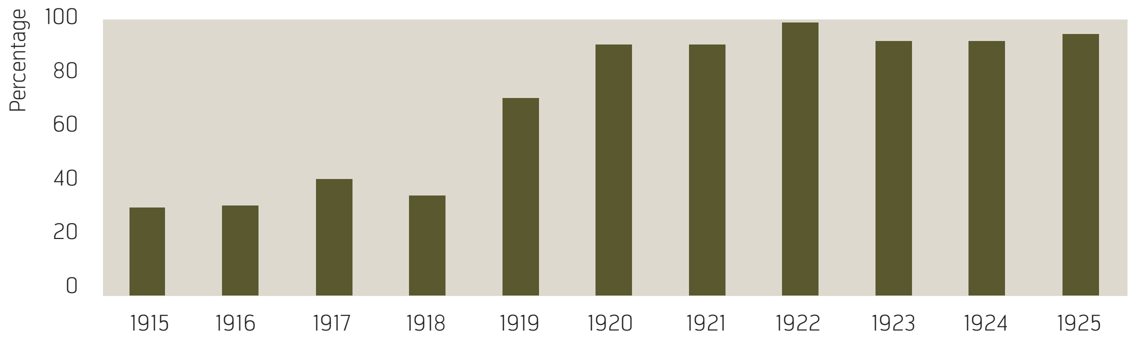
The figure below shows that the automatisation costs were completely dominating the other investment costs at the beginning of the 1920s.

Because of the very high investments, it was extremely important to have traffic dimensioning models which were fitted to the observed traffic. The very high investments in telephone exchanges underline the importance of having available traffic models and dimensioning tools.

Development of the Queuing Theory and the Teletraffic Theory

The Start of Work on Queuing Theory

Fr. Johannsen was appointed managing director of Copenhagen Telephone Company in 1903. He realized that the manual switches were not dimensioned in the right way. He published some pioneering work on this subject where he used the mathematical theory of probability (Johannsen, 1908 and 1910-11). From an economic point of view he stated:



Proportion of costs of the automation compared with total costs 1915 – 1925 (Bestorp, 1990)

The overloading of the subscribers resulted in considerable extra expenses on account of the telephone operators having to make repeated attempts to establish a connection.

A comparison of the increased cost due to an extra line and the reduced costs of the telephone operators, since attempts were needed to establish a connection for different numbers of lines, gave a sound foundation for how many lines the company should require for the single subscriber. (Jensen, 1996)

To be able to develop more precise dimensioning Fr. Johannsen established a scientific laboratory where he engaged the mathematician Agner Krarup Erlang. Erlang started to work on the holding times in a telephone switch (Erlang, 1909) and he identified that the number of telephone conversations which satisfied a Poisson distribution as well as the telephone holding time was exponential distributed.

Tore Olaus Engset (1865 – 1943)

The next important step in the queuing theory was done by the Norwegian Tore Olaus Engset.

The life of Tore Olaus Engset is very well described in the book *Tore Olaus Engset – The man behind the formula* (Myskja, 2002). He was for a long period Head of the Administrative Department in Telegrafverket, which also contained a traffic unit. The position was directly below the General Director (DG). He also functioned as DG for a period in the 1920s. In 1930, the Government for the first time appointed a person inside Telegrafverket as General Director and the most natural choice was Tore Olaus Engset. He held that position until he retired in 1935. He was also honoured with the Commander of Second Order of Dannebrog and Knight of the Legion of Honour (Myskja, 2002).

Tore Olaus Engset (1865 – 1943)

It is very understandable that the primary work of this man during normal working hours was not queuing theory and traffic dimensioning models. As pointed out both in (Natvig, 2000) and in (Myskja, 2002), *"The work of Tore Olaus Engset is extremely impressive, partly because it was carried out in late night hours outside the traditional working hours and also because he did not have access to our modern mathematical theory of probability and numerical methods organised for computers"*.

As described in this paper traffic in the telephone systems grew significantly from 1910 onwards in Norway, and especially in Kristiania, causing overflow and blocking of calls in the networks. Hence, it was extremely important to make the right plans for extending the networks based on the available grants from the Government. Tore Olaus Engset's work was fundamental. He developed queuing models or teletraffic loss models for finite sources which could be used for dimensioning manual, semi-automatic and automatic telephone exchanges.

His approach to the dimensioning is extremely elegant! The documentation of the work is done through an unpublished 128 page report in 1915 – which was recovered by Villy Bæk Iversen in 1996 (Iversen, 1996). Engset also published his methodology in 1918 (Engset, 1918). His elegant approach, valid for many queuing systems, is mainly based on combinatorial modelling. He calculates the probabilities for having i lines in an exchange occupied, given statistical equilibrium, based on individual inter-arrival times and individual holding times for each of the N subscribers in the access area. The model is based on using all permutations in drawing i individual subscribers out of N. Then, Engset calculates the loss probability as a function of different sizes of K, the number of lines in the exchange. Of course, it is not very practical to do dimensioning based on different traffic characteristics of each subscriber, but it is very convenient to do so for different groups of subscribers where subscribers in the different groups have different 'traffic' behaviour.

Engset's methodology produces surprisingly general results, which up to this day are perfectly applicable in queuing theory (ITU, 2005), even if the methodology is not based on the traditional approaches used in queuing theory! Villy Bæk Iversen underlines that the model is insensitive to both inter-arrival distribution and the holding time distribution (Iversen, 1996).

A simplification of Engset's general model gives the well known Engset distribution, which is a truncated Binomial model. Here it is assumed that all subscribers have identical inter-arrival distribution with parameter λ, and all subscribers have identical service time distribution with the mean $1/\mu$. Let K be the number of lines and N the number of subscribers. Then Engset's distribution, where $p(i)$ is the probability for i occupied lines, is given by:

$$p(i) = \frac{\binom{N}{i}\left(\frac{\lambda}{\mu}\right)^i}{\sum_{k=1}^{K}\binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k}$$

It should be noted that the famous Erlang B formula is a further simplification of Engset's distribution. Here, the number of sources is considered to be infinite, which of course is an approximation. Engset's simplified model is more precise because it assumes that the inter-arrival intensity, when i lines are occupied by the subscribers is $(M - i)\lambda$, while Erlang assumes that inter-arrival intensity is independent of the number of subscribers being serviced in the exchange.

Agner Krarup Erlang (1878 – 1929)

Agner Krarup Erlang (1878 – 1929)

Agner Krarup Erlang's life is well documented in (Brockmeyer, 1948). He finished his studies at the University of Copenhagen in 1901 acquiring the degree of candidatus magisterii (MA) with mathematics as principle subject. In 1908 the Copenhagen Telephone Company engaged Erlang. As pointed out he started to examine the holding times and published his first results in 1909 (Erlang, 1909).

Then in 1917, he published his most important work (Erlang, 1917). In section 1-7 he develops his famous Erlang B loss formula. As pointed out, the solution is based on considering equality of traffic streams through a cut between states.

Both Erlang and Engset have earlier been criticised when it comes to the validity of their models because they did not assume exponential holding time distributions. However, this criticism has been showed to be wrong, because the models are valid for different holding time distributions as long as there is a distinct mean (ITU, 2005).

Agner Krarup Erlang made a set of additional publications on teletraffic models in the 1920s and his famous loss formula, which was very applicable, got extremely popular for traffic engineers.

Erlang and Engset

This paper describes the incitements for developing teletraffic models. The telephone penetrations in the Scandinavian countries and Switzerland were significantly higher than in all other European countries at the start of the 20th century. However, from 1910 onwards the situation in Norway, and especially in Kristiania changed because of traffic congestions, a lot of complaints from the subscribers, limited investment grants, waiting lists for getting a telephone subscription etc. As mentioned earlier waiting lists were established – 5,000 potential subscribers were on the list in Copenhagen in 1920 and 6,000 in Kristiania. *This situation was the backdrop for the real start of development of the queuing theory. And the pioneers were Agner Krarup Erlang and Tore Olaus Engset!*

Erlang was employed by the Copenhagen Telephone Company in 1908 and Engset was already in 1894 Head of Traffic and Operations in Telegrafverket.

In 1910 Telegrafverket appointed a three man committee with Engset as one of the members to consider modernisation of the manual telephone systems in Kristiania by studying semi- and fully automatic telephone systems. The recommendation by the committee was finalised in 1913. During this period Engset had visited the Copenhagen Telephone Company and been aquainted with Erlang and P.V. Christensen who were active in traffic engineering. Hence, there are reasons to believe that Erlang and Engset exchanged views and knowledge on traffic modelling and dimensioning of the exchanges.

However, it is astonishing to realise that Erlang and Engset developed completely different approaches to calculating loss probabilities and dimensioning. Both methods are excellent and history has shown that the methods are still ‘future proof’ (ITU, 2005).

We now know that Erlang’s blocking formula is a simplification of Engset’s model. A natural question is of course: Why is Erlang’s work so well known and Engset’s work is not?

Engset had for a long period developed his dimensioning method, which he documented (128 pages) in 1915. He sent the documentation to Copenhagen Telephone Company and probably also to Stockholm (Iversen, 1996). However, time elapsed and he did not get his work published until 1918 (Engset, 1918). The question is of course – why?

Engset was a very busy man. His main work was not teletraffic and queuing theory – even if these aspects were very important. The main thing for him could be only to develop the results, like Fermat did, and not use too much energy to publish his results.

Another reason could be the comments he got from Erlang via Fr. Johannsen, which made him aware of the fact that his formula is just an approximation to the model. Engset even quotes this as a footnote in his paper (Engset, 1918), (Jensen, 1992). The discussion was about offered traffic and carried traffic based on Engset’s assumption regarding the observed traffic which he handles in the first part of his paper. This is a more general aspect, how to interpret the measurements and fit them to the modelling. The same arguments are also valid on Erlang’s loss formula!

Exchange of Information

Nowadays it is easy to access scientific information. There is a number of sources like traditional text books, university courses, journals, libraries, search engines like Google, tailored conferences and of course, establishment of personal networks.

Search and exchange of information is carried out rapidly by using Internet and e-mail. Therefore, it is difficult to understand the situation 300 years ago or even 100 years ago when information was sent by letter. The most crucial point in old times must have been the accessibility to research and scientific information and especially information between different scientific areas.

Going back to (Struyck, 1716) and his solution of the Gambler’s Ruin Problem, it is documented in this paper that he found the solution based on the same difference equations which 200 years later are used for solving the queuing system M/M/1. Engset used a completely different approach to solve the queuing systems, while Erlang used a similar approach. As mentioned in this paper, Erlang uses $i\mu$ instead of μ because of several servers instead of one. However, it must be pointed out that, while one thing is to solve the difference equations, another important thing is to deduce the equations itself.

Now, it is recognized that Erlang’s B formula was immediately used by traffic engineers, while Engset’s more general formula seems not to have been applied the first years. The main reason is probably that his solution simply was not known. Another reason could be that the formula is more complicated. The Engset loss formula admits the subscribers to have individual inter-arrival time distributions and holding time distributions. However, the simplification of his model assuming that all the subscribers have the same inter-arrival time and holding time distribution includes an additional parameter – the number of subscribers in the area compared to Erlang’s B formula. Hence, the table becomes larger. At that time, there were no computers available for the calculations, so all relevant tables had to be produced ‘by hand’.

Epilogue

This paper has briefly looked at the start of the mathematical theory of probability, the invention of the telephone and the start of the teletraffic/queuing theory. The following points have been drawn to attention:

Gerolamo Cardano was the real inventor of the mathematical theory of probability. However, he was not allowed by the Inquisition to publish his important work *Liber de Ludo Aleae – The Book on Games of Chance*. His work on probabilities was published in 1663, 87 years after his death, and inside a rather large ten cover volume of Cardano’s publications.

Hence, Blaise Pascal and Pierre de Fermat are considered by many to be the real founders of the mathematical theory of probability. The exchange of letters between Pascal and Fermat in 1654 was known and had impact on the future evolution of the probability theory!

Andrew Wiles isolated himself for seven years to prove Fermat’s last theorem. He published his proof, but there turned out to be a ‘hole’ in the proof. Happily, nobody managed to utilise the knowledge and Andrew Wiles completed the proof one year later!

Alexander Graham Bell has been considered to be the inventor of the telephone. This is not true. The Italian Antonio Meucci invented the telephone nearly 40 years before. He even tried to introduce the telephone and have it patented in the American market, but did not succeed.

Then, Alexander Graham Bell obtained the patent and utilised all economic and commercial possibilities in an excellent way. Antonio Meucci did not get anything – except that the American Congress 113 years after his death declared that he was the real inventor of the telephone!

Agner Krarup Erlang and Tore Olaus Engset were the real founders of the teletraffic/queuing theory. Erlang developed his famous B loss formula in 1917. Engset developed a more general loss formula in 1915 based on a completely different approach, which also was sent to Erlang's company. Erlang's model is a radical simplification of Engset's model.

Erlang became very famous for his work, and his loss formula has been widely used by teletraffic engineers. Engset's work was rather unknown for a long period and has recently been appreciated.

History is complicated, but it shows that some are lucky and succeed, while others do not. Sometimes there are a lot of random elements which affect evolution. But, there may also be other factors. Regarding Pierre de Fermat and Tore Olaus Engset, they held important positions in society which at that time were given a higher priority than their more theoretical work.

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